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The Optimum Structure for Government Debt[1]

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Abstract: *This paper studies the structural differences between implicit and explicit government debt in a two-generations-overlapping model with stochastic factor-prices. If a government can issue safe bonds and new claims to wage-indexed social security to service a given initial obligation, there exists a set of Pareto-efficient ways to do so. This set is characterized by the conflicting interests of the current young and the yet unborn generations regarding the allocation of factor-price risks.*
However, it is shown that there will always exist a simple intertemporal compensation mechanism which allows to reconcile these conflicting interests. This compensation mechanism narrows the set of Pareto-efficient debt structures until only one remains. This result hinges on the double-incomplete markets structure of stochastic OLG models where households can neither trade consumption loans nor factor-price risks privately.

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1 Introduction

Privatizing social security has often been described as a pure "shell game", where an implicit liability is replaced by an explicit liability of equal size.[2] From a different perspective, this equivalence between implicit and explicit government debt, may also be seen as a counterpart to the Modigliani-Miller Theorem in corporate finance. The underlying argument for this irrelevance result has its roots in the consumption loan nature of both debt instruments. A pure reallocation of resources between two adjacent cohorts can at most yield the biological interest rate.[3] For a deterministic economy, which is dynamically efficient in the sense of Diamond (1965), bonds are issued with a rate of return that is, at first sight, superior to the biological return earned on social security contributions. However, to prevent an eventual default, the government has to collect a tax that exactly offsets this return advantage. Taking these taxes into account, both instruments yield identical allocations.[4] In particular, they reduce long-run utility by crowding-out capital.

In stochastic overlapping generations models Enders and Lapan (1982), Merton (1983), Gordon and Varian (1988), Gale (1990), Krüger and Kubler (2005) and Gottardi and Kubler (2008) have shown that intergenerational transfers via PAYGO pension schemes and safe government debt may serve a second role. They allow to facilitate intergenerational risk sharing.[5] In-turn, these beneficial aspects of government debt have been

[2]See e.g. Breyer (1989), Fenge (1995), Belan and Pestieau (1999), Friedman (1999). See Sinn (2000) for a survey. Samuelson (1975) proves the related result that fully funded social security is also neutral. More recently, Ludwig and Reiter (2009) have extended the result to a stochastic setting with state dependent taxes.

[3]Samuelson (1958, 1959), Lerner (1959), Aaron (1966) and Cass and Yaari (1966). In the sequel, we abstract from technological progress as it does not change the basic tradeoffs.

[4]Both schemes pay the same returns, cause (in absence of intragenerational redistribution) the same excess burdens in the labor market, reallocate the same amount of resources between generations, displace an equal amount of private savings and lower long-run utility.

[5]In particular, Fischer (1983) and Gale (1990), discuss the desirability of safe debt and its maturity structure in an OLG context with rate-of-return risk. Enders and Lapan (1982) examine a mature pay-go scheme in an economy where fiat money is the only alternative store of value. Merton (1983) derives closed-form solutions for a three period OLG model with simultaneous demographic, TFP and income share risks. He shows that a tax and transfer system may replicate an (incomplete markets) equilibrium where agents can trade human capital freely. In the Merton (1983) setting such an intervention is always warranted as young agents would starve under "total market failure". Bohn (1998, 2003) shows that a constant debt to GDP ratio leads to pro-cyclical debt issues, that amplify aggregate risks. Starting from a situation without government debt, Krüger and Kubler (2005) give numerical evidence that the introduction of unfunded social security is unlikely Pareto-improving − despite its risk sharing capacities − due to the crowding-out of capital. Gottardi and Kubler (2008) discuss the prospects of an ex-ante Pareto-improving introduction of unfunded social security in an economy with land. See Diamond (1977, 2000) for a broader assessment of intergenerational and intragenerational insurance aspects of social security, and Shiller (1999) for more references on the sharing of aggregate risks. See Abel (2001), Diamond and Geanakoplos (2003), and Ball and Mankiw (2007) for different approaches to utilize trust-fund assets − a question somewhat related to the present one. To focus firmly on the unfunded component of social security we will not introduce a trust-fund. Moreover, we leave-out ideosyncratic risks. As Bester (1984) and Abel (1989) show these risks can be insured within each cohort, i.e. they are not essential in the current context.

compared to the negative long-run losses which stem from the crowding-out of capital. In particular Green (1977), Krüger and Kubler (2005) and Gottardi and Kubler (2008) examine this trade-off between risk sharing and worsening factor-prices. Their analysis indicates that even the introduction of a very small social security system tends to decrease long-run utility. That is, the positive risk sharing effect is dominated by the negative crowding-out effect.[6]

The current analysis complements this literature by taking a different perspective. We ask whether it is possible to restructure the vast debt which is already present in most countries in a Pareto-improving manner. Following this question, we show that it is possible to separate the crowding-out effect from the risk sharing problem. Changes in the composition of the public debt leave *expected* intergenerational transfers constant over time but alter the allocation of factor-price risks between different cohorts. Changes in the size of the debt change intergenerational transfers but tend to leave the allocation of factor-price risks unaltered. This *separation* of crowding-out and intergenerational risk sharing associated with public debt will in general allow the government to make a restructuring of the debt Pareto-improving.

To derive this result, we set up an initial value problem. Each member of an initial old generation holds claims from past pension promises and debt issues amounting to g_0. The government can now raise a share λ of the revenue needed to service these claims through the introduction of a linear social security tax on the current young generations wage income. The remainder share $1-\lambda$ has to be financed by selling safe debt. Finally, there is the group of yet unborn generations who have to service future pension claims issued to the current young generation.

There are two corollaries to the separation result sketched earlier: (i) if the government can only change the composition of the existing debt, there will be a set of efficient debt structures and another set of inefficient ones. The efficient set is characterized by the conflicting interests of those agents who are currently young and those who are yet unborn. The unborn generations benefit from the ex-ante diversification of their wage risk if a large share λ of the initial debt is injected into social security. The current young, who have already observed their wage income, on the contrary prefer safe debt, i.e. safe retirement benefits. (ii) if the government can also issue/recover additional bonds, i.e. change the size of the expected future intergenerational transfers, the set described in (i) can be narrowed to a Pareto-optimal debt structure, which maximizes societies (ex-ante) "Marshallian surplus" from intergenerational risk sharing. Put differently, the government can use its two instruments, i.e. the size and the composition of the debt, to steer the economy towards one point on the contract curve.

This second result appears to be of particular interest, when compared to the problem

[6] Intuitively this result is plausible if we think of it in terms of the Finetti (1952), Pratt (1964), Arrow (1970) approximation: $E[U(c_0+\varepsilon)] \approx U(c_0)+U'(c_0)\mu_\varepsilon+\frac{1}{2}U''(c_0)\sigma_\varepsilon^2$. The crowding-out of capital induces first order welfare losses by lowering expected consumption μ_ε. The risk sharing benefits, however, are only of second order. For the above approximation we have used the approximation $\sigma_\varepsilon^2 = E[\varepsilon^2]-E[\varepsilon]^2 \approx E[\varepsilon^2]$, which is accurate if $E[\varepsilon]$ is small. For $E[\varepsilon]=0$ we have $E[U(c_0+\varepsilon)] \approx U(c_0)+\frac{1}{2}U''(c_0)\sigma_\varepsilon^2$. In this case, the lower consumption would be associated with a reduction in c_0.

of optimal capital accumulation in a deterministic Diamond (1965) model. In analogy to our result (i), there always exists a set of efficient capital intensities. This means that every change in the capital intensity requires a welfare criterion as we can either shift resources into the future or redirect resources from the future towards current generations.[7] In the present stochastic setting, however, we show that it is possible to compensate intertemporally. We can shift resources and risks between the current young and the yet unborn members of society *simultaneously* and *independently*. As a consequence, the government can compensate intertemporally and narrow the set of efficient debt structures (without compensation) to the set of points on the contract curve (with compensation).

Regarding our assumptions, a notable aspect of our analysis is that we rule-out state-contingent lump-sum transfers. Following Merton (1983), Gordon and Varian (1988), Bohn (1998, 2003), Krüger and Kubler (2005) and Gottardi and Kubler (2008) we try to capture the basic features of most real-world pension and debt schemes by limiting the government debt instruments to safe bonds and a linear social security contribution rate on wages. We do so for two reasons: (*i*) while state-contingent lump-sum transfers may allow to reach better allocations than our simplistic debt instruments, they are not observed in actual policy. (*ii*) The optimal allocations which are derived for such state-contingent tax and transfer systems usually imply that the public debt follows a random walk as described in Gordon and Varian (1988) and Ball and Mankiw (2001, 2007).[8] Hence, if the government would actually implement these policies, it would default in finite time with probability one. One may therefore argue that such a risk sharing policy amplifies rather than dampens the small risks faced by each generation as they create a tremendous default risk.

Subsequently, in Section 2 we begin by laying out our model. The representative households, are assumed to maximize expected utility. Moreover, first and second period consumption are assumed to be normal goods. Savings can be invested in a risky and a safe production technology. Wages are determined according to a third risky technology. As in Diamond and Geanakoplos (2003), it is assumed that aggregate investment does not affect marginal returns. This tri-linear setting will help us to bring out the underlying economic mechanisms more clearly.[9] In a different interpretation we may think of our model as a small open economy. Subsequently, the budget constraints of the social security system

[7]The lack of such a compensation mechanism led to the turnpike literature, see, e.g., Samuelson (1968) or Blanchard and Fischer (1989). The absence of such an intertemporal compensation mechanism is of course also the reason for the intertemporal efficiency of pay-go schemes that we have been referring to in Footnote 2.

[8]Gordon and Varian (1988), p. 192, and Ball and Mankiw (2001, 2007) (Proposition 2), point out that their debt schemes that reallocate risks "optimally" imply that per capita debt will follow a random walk. Hence per-capita debt will hit any boundary in finite time. Consequently, as Gordon and Varian (1988), p. 192 point out, the economies total assets will eventually be negative, forcing the government to default at some point.

[9]As the per capita size of *expected* intergenerational transfers will be kept constant over time we do not expect large changes in aggregate savings once implicit debt is replaced by explicit debt (cf. Diamond (1996)). Hence the crowding-out effects along the neoclassical competitive factor-price-frontier, which are so notable when *additional* debt is issued, do not come into play in the current analysis.

and the treasury are introduced. With the model in place, the two main results (*i*) and (*ii*) are derived in Section 2. In Section 3, we show that our results carry over once some of the restrictive assumptions made in Section 2 are relaxed. Namely, the assumption of a constant risk-free rate will be dropped. Moreover, we consider a defined benefit social security system, and briefly touch upon an economy with intra-cohort heterogeneity. Section 4 offers concluding remarks.

2 The Model

In this section we first introduce our assumptions regarding technology and preferences. Subsequently, we trace out the preferences of the current young and the yet unborn generations regarding the composition of the debt. Section 2.5 contains the key results on the separability of crowding-out and risk-sharing.

2.1 Population and factor-prices

The economy is inhabited by two-period-lived agents that form overlapping generations. During the first period of life each agent supplies one unit of labor inelastically. Population evolves according to:

$$N_{t+1} = (1+n)N_t, \tag{1}$$

where N_t is the size of the cohort born in period t and $1+n$ is the number of children raised by each member of cohort t.

The wage rate w_t and the interest rate to risky capital R_t are both stochastic. They follow an exogenously given, serially i.i.d., distribution. The stochastic wage rate w_t realized in period t has a lower bound $\check{w} > 0$. Risky investments have the limited liability property, i.e. $\check{R}_t \geqq -1$. Furthermore the rate of return R_t may be correlated with the wage rate w_t, i.e. $cov(w_t, R_t) \gtreqless 0$. In our baseline specification we assume that the safe rate r is exogenously given; respectively defined by a safe linear technology. In the sequel we also assume that $\check{R} < r < E[R]$, such that both risky and riskfree assets may be held by risk-avers investors. In Section 3, we relax the assumption of a constant riskfree rate.

2.2 Implicit and Explicit Government Debt

The government can interact with the competitive economy both via an unfunded pay-as-you-go social security system and through the intertemporal budget constraint of the treasury. While both of these schemes may be used to roll over debt, they differ with respect to the way that wage-income is taxed.

An unfunded social security system with a contribution rate τ^s and per capita benefits p is characterized by its budget constraint:

$$\tau_t^s w_t N_t = p_t N_{t-1}. \tag{2}$$

Using the biological interest rate relation (1), constraint (2) can be rewritten, such that per capita pension benefits are given by:

$$p_t = (1+n)\tau_t^s w_t. \tag{3}$$

Equation (3) indicates that an agent born in period t will contribute an amount $\tau^s w_t$ to the pension system in exchange for uncertain future benefits $(1+n)\tau_t^s w_{t+1}$. In terms of expectations, the consumption loan scheme will grow at rate n if the contribution rate is fixed. In this case, the per capita size of expected transfer remains constant:

$$E_{w_{t+1}}[p_{t+1}] = (1+n)\tau^s E_{w_{t+1}}[w_{t+1}]. \tag{4}$$

The second channel through which the government can roll over debt is the treasury's budget constraint. Denoting the total amount of outstanding debt by B_t, the amount of claims that are due in period $t+1$ by B_{t+1} and the treasury's tax rate by τ_t^t, the treasury's intertemporal budget constraint for period t is:

$$B_{t+1} = (1+r_{t+1})(B_t - N_t\tau_t^t w_t). \tag{5}$$

Defining debt per worker by $b_t \equiv \frac{B_t}{N_t}$ and substituting (1) into (5) yields:

$$(1+n)b_{t+1} = (1+r_{t+1})(b_t - \tau_t^t w_t). \tag{6}$$

If no taxes were levied, per capita debt would grow at a proportional rate of $\frac{r_{t+1}-n}{(1+n)}$, from period t to period $t+1$. To ensure that in per capita terms no additional debt is passed forward from generation t to generation $t+1$, the treasury has to collect taxes from generation t amounting to:

$$\tau_t^t w_t = \frac{r_{t+1}-n}{(1+r_{t+1})} b_t. \tag{7}$$

Taxes are either positive or negative depending on whether the returns to intergenerational redistribution dominate market returns, i.e. if $r \gtreqless n$.[10]

2.3 The Structure of Government Debt

At the beginning of time there is an initial generation -1 of retirees and a generation 0 of workers. The generation of retirees holds per capita claims to an existing social security system and/or from past issues of government debt, amounting to g_0. To service these claims the government has to raise a revenue of $\frac{g_0}{1+n}$ from each member of generation 0. A

[10]The taxes needed to keep per capita debt from growing to infinity, will be paid by the young consumers. However, as long as the representative agent invests into the riskfree technology, he will be indifferent between a tax of $\frac{(r_{t+1}-n)}{1+r_{t+1}}b$ when young or a tax of $(r_{t+1}-n)b$ when old.

share $\lambda \in [0,1]$ of the needed revenue can now be raised via the initiation of an unfunded pension scheme with a defined contribution rate τ^s:[11]

$$\tau_0^s w_0 N_0 = \lambda g_0 N_{-1}, \quad \Leftrightarrow \quad \tau^s = \tau_0^s = \frac{\lambda}{w_0} \frac{g_0}{(1+n)}. \tag{8}$$

The remainder share $(1-\lambda)$ can then be raised by issuing safe government bonds:

$$(1-\lambda) g_0 N_{-1} = B_0, \quad \Leftrightarrow \quad (1-\lambda)\frac{g_0}{(1+n)} = b_0. \tag{9}$$

Recalling (7), per capita taxes in period 0 must satisfy:

$$\tau_0^t = (1-\lambda)\frac{(r_1 - n)}{(1+r_1)w_0}\frac{g_0}{(1+n)}. \tag{10}$$

Once we do not ask any future generation to redeem the debt, all subsequent generations will be taxed according to:

$$\tau_t^t = (1-\lambda)\frac{r_{t+1} - n}{(1+r_{t+1})w_t}\frac{g_0}{(1+n)}. \tag{11}$$

Inspection of (8) and (11) immediately yields the equivalence proposition that we have been referring to in the introduction.[12] In what follows, we drop the time index where no misunderstanding is expected.

2.4 The Optimum Structure for Government Debt

In this section we start by tracing out the preferences of the current young regarding the structure for government debt λ. Subsequently, we characterize the interests of the yet unborn generations. With these results at hand, the two main results are derived in Section 2.5. A representative member of cohort 0 can allocate his net income to first period consumption c^1, invest an amount a_0 into the safe technology and devote h_0 to the risky technology:

$$\begin{aligned} \max_{c^1, c^2} \quad & W = U(c^1) + \beta E_{wR}[U(c^2)]; \quad U'() > 0, \quad U''() < 0, \\ s.t. \quad & c^1 = w_0(1 - \tau_0^t - \tau_0^s) - a_0 - h_0, \\ & c^2 = a_0(1+r) + h_0(1+R_1) + \tau_0^s w_1 (1+n). \end{aligned} \tag{12}$$

[11] Note that as with the explicit debt scheme, the amount resources transferred via social security may not permanently outpace the economy. At the same time lowering the contribution rate would amount to a repayment of some debt by the affected generation of retirees. To make both schemes feasible and comparable, we therefore fix τ^s.

[12] In the standard Diamond (1965) economy, the steady state budget constraint of the representative agent reads $c^1 + \frac{c^2}{1+r} = w(1-\tau^s-\tau^t) + \frac{\tau^s w}{1+r}(1+n)$. Plugging the two budget constraints of the treasury (11) and the social security administration (8), with $w_0 = w$, into this budget constraint yields for the right-hand-side: $w - \frac{r-n}{1+r}\frac{g_0}{1+n}(1-\lambda) - \frac{g_0}{(1+n)}\lambda + \lambda\frac{g_0}{1+r} = w - \frac{(r-n)g_0}{(1+r)(1+n)}$. The life-cycle savings condition is also independent of λ: $(1+n)(\lambda\frac{g_0}{1+n} + (1-\lambda)\frac{g_0}{1+n} + k) = g_0 + (1+n)k = s$. Hence, changing the debt structure along the steady state, is irrelevant as it neither affects the household's budget constraint nor the life-cycle savings condition.

The corresponding first order conditions, which imply a_0^* and h_0^*, are:

$$\frac{\partial W}{\partial a_0} = -U'(c^1) + \beta(1+r)E_{wR}[U'(c^2)] = 0, \tag{13}$$

$$\frac{\partial W}{\partial h_0} = -U'(c^1) + \beta E_{wR}[(1+R)U'(c^2)] = 0. \tag{14}$$

If felicity, $U()$ in (12), is such that first and second period consumption are normal goods we have:[13]

$$s = s(w;\tau^s) = a + h; \quad 0 < \frac{\partial s}{\partial w} < (1-\tau^s). \tag{15}$$

Equipped with these conditions, the social planner can, disregarding the utility of subsequent generations for the moment, use the two debt instruments by choosing λ such that the indirect utility of generation 0 is maximized. Taking into account the budget constraints (8) and (10) yields the planning problem:[14]

$$\begin{aligned} \max_{\lambda} V_0 = \quad & U(w_0(1-\tau_0^s-\tau_0^t) - a_0 - h_0) \\ & +\beta E_{wR}[U(a_0(1+r) + h_0(1+R) + \tau_0^s w(1+n))], \\ s.t. \quad & (8), \quad (10). \end{aligned} \tag{16}$$

Utilizing the envelope condition (13) and the covariance rule, λ^* is implicitly defined by:

$$\frac{dV_0}{d\lambda} = \frac{U'(c^1)g_0}{1+r}\Big(\frac{E[w]-w_0}{w_0} + \frac{cov_{wR}(U'(c^2),w_1)}{w_0 E_{wR}[U'(c^2)]}\Big) = 0. \tag{17}$$

Condition (17), which is reminiscent of the C-CAPM, indicates that members of generation 0 will benefit from a high fraction of debt that is injected into the social security system as long as the expected excess rate-of-return on this fraction of debt, compared to the after-tax-return on safe bonds, is positive, i.e. $\frac{Ew-w_0}{w_0} > 0$. The other relevant component is the covariance between second period marginal utility and the pension benefit. Depending on $cov(R_1, w_1) \gtreqless 0$, we have $cov(U'(c^2), w_1)_{|\lambda=0} \lesseqgtr 0$, i.e. the wage-indexed social security claims may or may not be a welcome opportunity to diversify stock market risks.

[13] The increment in income from a high realization of w_t is given by $(1-\tau^s-\tau^t(w_t)) + \frac{\partial \tau^t(w_t)}{\partial w_t} w_t = (1-\tau^s)$.

[14] Note that there is no life-cycle savings condition for bonds and capital in a small open economy, i.e. we only take note of the taxes that are needed to keep per capita debt from growing. In a closed economy with a tri-linear technology, we can also neglect the market clearing condition as long as agents demand safe investments in excess of the debt offered. In the following we assume that agents are equating at the margin, i.e. we omit the prospect of Kuhn-Tucker-type ramifications.

1) Subsequent Generations The social planner's perspective on the welfare of subsequent generations, which is obviously connected to the current choice of λ, will be an ex-ante perspective. While the social planner knows the distribution over R and w, the realizations are yet unknown. The agents, however, will start to make their consumption savings decisions in period t *after* w_t has been realized. The consumer's behavior is therefore still characterized by conditions (13) and (14) which imply the wage dependent investment decisions $a_t = a_t(w_t; \lambda)$ and $h_t = h_t(w_t; \lambda)$. Put differently, the social planner, who optimizes ex-ante utility, has to take note of the agent's investment decisions conditional on the realization of w_t. Moreover, the budget constraints (8) and (11) have to be satisfied in each period. From the perspective of period 0, the planning problem is therefore given by:

$$\max_{\lambda} V_t = E_{w_t}\Big[U(w_t(1-\lambda\frac{g_0}{w_0(1+n)}) - \frac{r-n}{(1+r)}\frac{(1-\lambda)g_0}{(1+n)} - a_t - h_t)\Big] \qquad (18)$$
$$+\beta E_{w_t w_{t+1} R_{t+1}}\Big[U(a_t(1+r) + h_t(1+R) + \lambda\frac{g_0}{w_0}w_{t+1})\Big].$$

The first order condition for an optimum debt structure, taking the envelope conditions (13) and (14) into account (see Appendix 5.1), is then given by:[15]

$$\frac{dV_t}{d\lambda} = \frac{g_0}{(1+n)}\Big(\frac{n-r}{1+r}\frac{E[w]-w_0}{w_0}E_{w_t}[U'(c^1)] \qquad (19)$$
$$-cov_{w_t}(U'(c^1), \frac{w_t}{w_0}) + \beta(1+n)cov_{w_t w_{t+1} R}(U'(c^2), \frac{w_{t+1}}{w_0})\Big) = 0.$$

Equation (19) characterizes the debt structure λ^{**} which maximizes long-run expected utility. Inspection of (19) indicates that agents who are not yet born will suffer a loss from excessive intergenerational redistribution if the safe returns exceed the biological returns on consumption loans. That is, the expected excess amount of resources – when compared to bonds which are not wage-indexed – that is redistributed via social security is given by $\frac{Ew-w_0}{w_0}$.[16] The second element is the intergenerational diversification of wage-income risk. With $\lambda > 0$ we have a positive social security tax rate τ^s, which transfers some of the risk associated with the realization of w_t into period $t+1$, where w_{t+1}, i.e. the pension benefits are realized. The sufficient condition for an interior optimum requires that $\frac{dV}{d\lambda}$ is downward-sloping in λ. A first inspection of (19) suggests $\frac{dcov(U'(c^1),w_t)}{d\lambda} > 0$, $\frac{dcov(U'(c^2),w_{t+1})}{d\lambda} < 0$, and therefore $\frac{d^2V}{d\lambda^2} < 0$. Hence, as we shift wage-income risk from the first into the second period, we expect the wage related covariance risk to move in the same direction (see Appendix 5.2 for the associated conditions). However, as the set of

[15] Taking advantage of our assumption that the stochastic wage rate w_t is serially uncorrelated we may rewrite $cov_{w_t w_{t+1} R}(U'(c^2), \frac{w_{t+1}}{w_0}) = cov_{w_{t+1} R}(E_{w_t}U'(c^2), \frac{w_{t+1}}{w_0})$. If such a serial correlation existed, it would affect the location of λ^{**}. If a and h are normal, we have $\frac{da}{dw_t} > 0$ and $\frac{dh}{dw_t} > 0$; thus we would have a smaller λ^{**} if $cov(w_t, w_{t+1}) > 0$, and vice versa.

[16] The expected intergenerational transfer through social security is $E[\tau^s w] = \frac{g_0}{(1+n)w_0}Ew$. Regarding bonds, the transfer is $\frac{g_0}{(1+n)}$. The difference in the expected size of the transfers, which yield the inferior biological return, is therefore given by $\frac{g_0}{(1+n)}\frac{(Ew-w_0)}{w_0}$.

admissible debt structures is closed and bounded, there will always exist a “best” debt structure $\lambda^{**} \in [0,1]$.

The efficiency of the *size* of the debt scheme can be assessed once we ask whether the unborn generations benefit from a larger initial debt. Taking the first derivative of V_t with respect to g_0 yields:

$$\frac{dV_t}{dg_0}_{|d\lambda=0} = \frac{n-r}{(1+r)(1+n)}\Big(\frac{w_0+\lambda(E[w]-w_0)}{w_0}\Big)E[U'(c^1)] \tag{20}$$
$$+\lambda\frac{1}{(1+n)}\Big((1+n)\beta cov(E_{w_t}[U'(c^2)],\frac{w_{t+1}}{w_0}) - cov(U'(c^1),\frac{w_t}{w_0})\Big) \gtreqless 0.$$

The first element in (20) is the familiar return condition; larger intergenerational reallocation of resources is desirable as long as consumption loans dominate market returns. The second element reflects the benefits from intergenerational risk sharing through the share λ of debt that is injected into the pension system. To see this more clearly, we recall (19) and rearrange (20) such that:

$$\frac{dV_t}{dg_0}_{|d\lambda=0} = \frac{n-r}{(1+r)(1+n)}E[U'(c^1)] + \frac{\lambda}{g_0}\frac{dV_t}{d\lambda} \gtreqless 0. \tag{21}$$

If λ is zero or at its long-run optimum λ^{**}, the second risk sharing related term vanishes and (21) exhibits the pure interest condition.

Furthermore, (21) indicates that safe debt does not reallocate risks, while social security does. This is the opposite of the Bohn (1998, 2003) conclusion, where debt was issued pro-cyclical such that it shifted risks towards future generations. Equation (21) also shows that if the national debt is small, then this debt should be injected entirely into the pension scheme if $\frac{dV_t}{d\lambda}, \frac{dV_0}{d\lambda} > 0$, such that the benefits from risk sharing are maximized with $\lambda = 1$. In a different interpretation, the sign of (21) is the subject studied by Green (1977), Krüger and Kubler (2005) and Gottardi and Kubler (2008).

2.5 Efficiency

Inspection of our above analysis indicates that generation 0 will prefer a debt structure λ^*, that is a solution to (17), rather than λ^{**}, which solves (19).[17] If the government can control the composition of the public debt only, all debt structures located between λ^* and λ^{**} are Pareto-efficient. Raising λ beyond λ^* will increase expected utility of all unborn generations at the expense of generation 0. Starting with λ^{**}, the same applies when λ is lowered. Hence, we have the following proposition:

Proposition 1. *If the government can only implement the debt structure that is used to roll over the initial debt, there exists a set* $[\lambda^*, \lambda^{**}] \subseteq [0,1]$ *of efficient financing methods.*

[17]For appropriate $(Ew - w_0, r-n, cov(w,R))$, λ^* may actually coincide with λ^{**}. In this case both generations prefer − though for different reasons − the same debt structure, and, except for choosing this structure, no additional government intervention is necessary. The same applies when corner solutions coincide.

This set is characterized by the conflicting interests of the current young and the yet unborn generations.

Diagrams 1a and 1b illustrate this trade-off. We now trace out the set of Pareto-improving

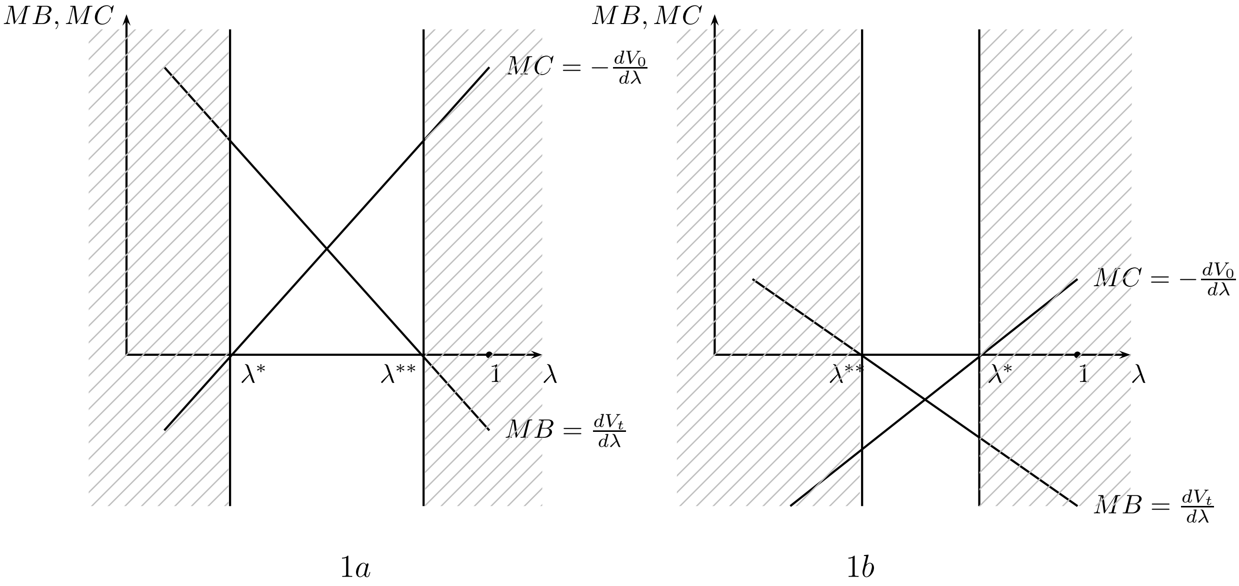

Diagram 1: *Efficient debt structures.*
Diagrams 1a *and* 1b *illustrate the gains and losses of generation* 0 *and one representative member of the yet unborn generations. All debt structures located in the dashed area are inefficient. Diagram* 1b *depicts a situation that may occur if* $E[w] \gg w_0$ *and* $r \gg n$.

transitions from one debt scheme to another, which are available once the government can change both, the composition and the structure of the public debt. As we have stressed earlier, with these two instruments, it will be possible for the government to separate the risk sharing properties of the public debt from the crowding-out effect.

1) Efficiency with Government Intermediation Suppose now that the initial conditions are such that $\lambda = \lambda^* < \lambda^{**}$. In this case each member of the yet unborn generations is willing to accept a (slightly) higher level of public debt in exchange for a more favorable composition $\tilde{\lambda} > \lambda^*$ of the debt. At the same time members of the current young generation are willing to accept additional pension claims and safe bonds in exchange for the less favorable allocation of factor-price risks associated with $\tilde{\lambda}$. The government can now offer generation 0 to increase the per-capita (in terms of generation -1) size of the public debt by π. The new debt scheme has a per-capita (of generation 0) size of $\frac{g}{1+n} \equiv \frac{g_0+\pi}{1+n}$. The associated Lagrangian, which allows to trace-out the set of Pareto-improving pension reforms, is then given by:

$$\max_{\pi,\lambda,\mu} \mathcal{L} = V_0(\lambda,\pi) + \mu(V_t(\lambda,g) - \bar{V}); \quad V_t(\lambda,g) \geqq \bar{V} \equiv V_t(\lambda^*,g_0), \quad g \equiv g_0 + \pi. \qquad (22)$$

Where the Lagrangian (22) consists of the indirect utility functions of the current young and the yet unborn generations which where discussed earlier in Section 2.4. The additional argument π in V_0 reflects that members of generation 0 receive additional safe consumption (after taxes) amounting to $(1-\lambda)\frac{1}{1+r}\pi$ and additional pension claims $\lambda\frac{\pi}{w_0}w_1$ once the debt scheme is increased in size. The partial derivative $\frac{\partial V_0}{\partial \pi}$ is therefore positive. Regarding future generations, we focus on the interesting case where resources are scarce and an increase per-capita debt alone is not Pareto-improving. That is, the partial derivative $\frac{\partial V_t}{\partial g}$, described in (21), is assumed to be negative. Finally, as per-capita debt does not grow over time it is sufficient to represent future generations using only one lagrangian multiplier μ. Regarding the first order conditions associated with (22) we have:

$$\frac{\partial \mathcal{L}}{\partial \pi} = \frac{\partial V_0}{\partial \pi} + \mu \frac{\partial V_t}{\partial g} = 0, \tag{23}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{\partial V_0}{\partial \lambda} + \mu \frac{\partial V_t}{\partial \lambda} = 0. \tag{24}$$

Combining (23) and (24) we can drop the Lagrangian multiplier μ. The first order condition for the optimum structure for government debt λ^{***} is then:

$$\frac{\frac{\partial V_0}{\partial \lambda}}{\frac{\partial V_0}{\partial \pi}} = \frac{\frac{\partial V_t}{\partial \lambda}}{\frac{\partial V_t}{\partial g}}. \tag{25}$$

Condition (25) indicates that the optimum structure for government debt is associated with a point on the contract curve. It equalizes the marginal rates of substitution between the burden of an additional unit of debt and risk sharing benefits between current and future generations. By varying the size and composition of the debt it is possible to recover the efficiency gains displayed in Diagram 2 in a Pareto-improving manner. We therefore have the following proposition:

Proposition 2. *If the government can vary both, the size of the public debt and its composition, it is possible to separate the crowding-out effect from the risk sharing properties of the public debt scheme. The efficiency gains associated with the optimum structure for government debt λ^{***} can be recovered in a Pareto-improving manner.*

Remark 1*: The optimum structure for debt λ^{***} may be at a corner solution.*

Remark 2*: Different reference levels $\bar{V}_t$ for the utility of future generations will change the distribution of the efficiency gains brought about by the implementation of λ^{***}. The associated income effects will slightly affect the location of λ^{***}.*

Remark 3*: If the initial debt structure is such that $\lambda > \lambda^{***}$, some of the efficiency gains associated with the implementation of λ^{***} can be passed forward to compensate the unborn generations. In this case, generation 0 gives up resources in exchange for lower labor income risk.*

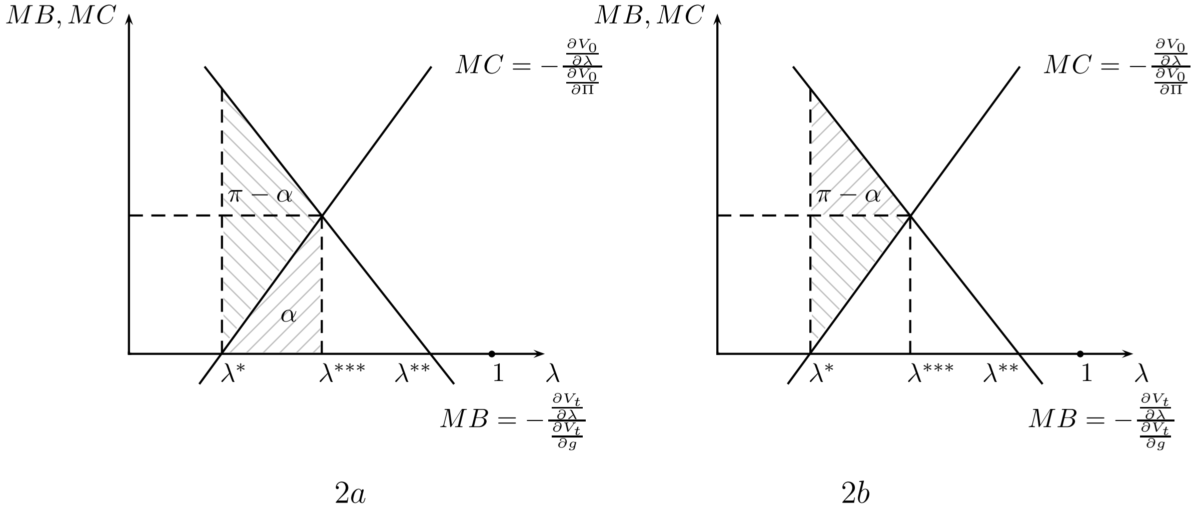

Diagram 2: *Efficiency gains from intertemporal compensation.*
Diagrams 2a and 2b illustrate the compensation described in (25). In the case where $\bar{U} = U_t(\lambda, g_0)$ *all efficiency gains* $\pi - \alpha$ *accrue to Generation* 0.

Remark 4: *To keep in touch with the steady state as a reference point, Proposition 2 neglects the possibility of a repeated restructuring of the debt.*

Remark 5: *The golden rule of accumulation lends itself to the interpretation: maintaining a capital intensity that permanently exceeds the golden rule level is inefficient. In the present case we have a stronger result: maintaining any debt structure that permanently differs from* λ^{***} *is inefficient.*

2) Interpretation At this point it is interesting to compare the present result on the possibility of Pareto-improving social security reforms with the earlier negative results by Green (1977), Krüger and Kubler (2005), Gottardi and Kubler (2008). In the case, where an initial debt is already present, a change in the composition of this debt reallocates factor-price risks but does not affect the size of the intergenerational transfer. By choosing λ^{***} as a debt structure it is now possible to tailor a particular exchange of risks and resources such that it is beneficial to both groups of agents. Namely, those living in the "long-run" and those who live today. Diagram 3 illustrates this. Curve 1 represents the long-run consequences of a linear social security tax. As this tax increases, the economy moves from the origin to a certain point e.g. K. Curve 2, which is steeper than 1, shows the threshold where future generations are indifferent between the crowding-out of capital and the risk sharing benefits. Finally, point C is an allocation that can be reached in the manner described above: a change in the composition of the debt reallocates many risks via the linear social security tax. The change in the allocation of resources is mainly due to the change in the size of the debt π. Put differently, by introducing a linear social security tax alone the government can only move along arrow 1. If there is already an

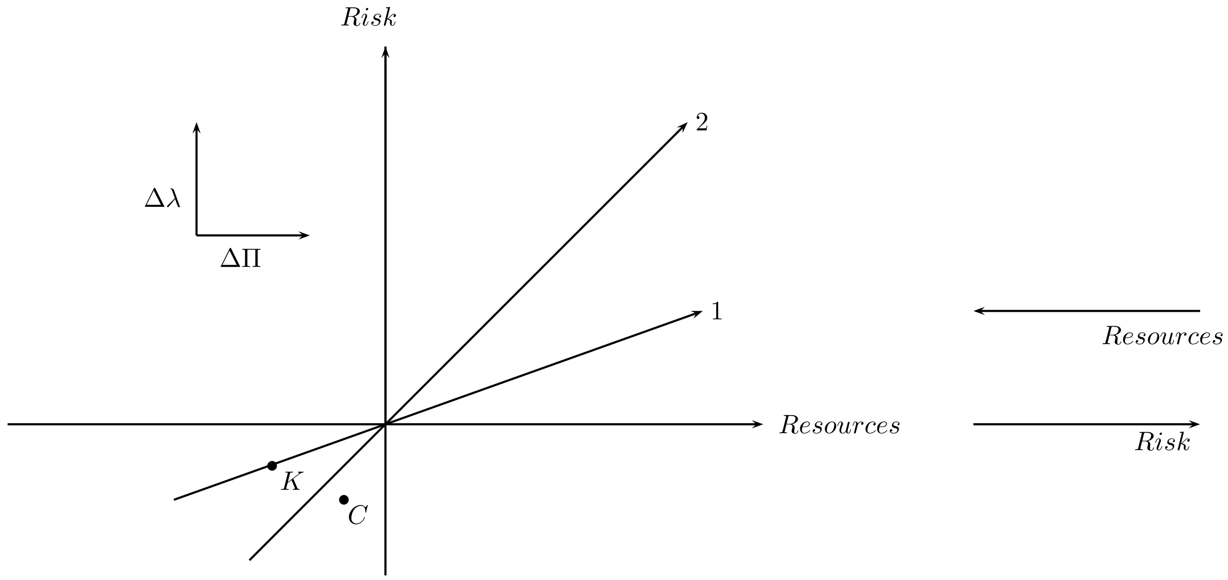

Diagram 3: *Separation of crowding-out and risk sharing*
A linear social security tax 1 *implies a combination of crowding-out and intergenerational reallocation of factor-price risks. Introducing a particular social security tax moves the economy from the origin to point K. Line* 2 *indicates the minimum reallocation of risks necessary to compensate future generations for the negative crowding-out effect. In the present case the government has two instruments available. It can therefore move freely in the risk-resource plane and implement the optimal allocation C.*

initial debt present it has two linearly independent instruments. In this case it can move in the entire plane, where point C is associated with an optimal pair λ^{***}, π^{***}.

3) Another Interpretation In a different interpretation (25) may be seen as an intertemporal version of the Samuelson (1954) condition for the efficient provision of a public good. Recalling equation (21) we can rewrite (25) such that:[18]

$$\begin{aligned} \frac{\frac{\partial V_0}{\partial \lambda}}{\frac{\partial V_0}{\partial \pi}} &= \frac{\frac{\partial V_t}{\partial \lambda}}{-E_{w_t}[U'(c^1)]\frac{r-n}{(1+r)(1+n)} + \frac{\lambda}{g}\frac{\partial V_t}{\partial \lambda}} \\ &= \sum_{t=1}^{\infty} \left(\frac{1+n}{1+r}\right)^t \frac{\frac{\partial V_t}{\partial \lambda}}{-E_{w_t}[U'(c^1)]\frac{1}{1+r} + \frac{\lambda}{g}\frac{\partial V_t}{\partial \lambda}\frac{1+n}{r-n}}. \end{aligned} \tag{26}$$

[18]For $r > n$, we have $\sum_{t=1}^{\infty} \left(\frac{1+n}{1+r}\right)^t = \frac{1+n}{r-n}$. Note that the RHS of condition (25) is the marginal rate of substitution between an increase in λ and and increase of the debt level of one unit. The new formulation in (26) is the sum of the marginal rates of substitution between a marginal increase of λ and a marginal increase in the tax level.

Condition (26) indicates that all future generations benefit from the public good "risk sharing" which is embodied in the debt scheme. The cost with the provision of this public good has to be incurred only once by generation 0, which bears additional wage-related risk. Depending on its position on the time axis, the present value of tax payments differs from cohort to cohort. The first element $-E[U'(c^1)]\frac{1}{1+r}$ in the numerator of the marginal rate of substitution of future generations indicates the negative crowding-out effect. The second element $\frac{\lambda}{g}\frac{\partial V_t}{\partial \lambda}$ is positive. As a share λ of the new debt π is injected into social security. This increases the willingness of future generations to accept a higher level of public debt.

The analogy to the problem of public good provision also extends to the aspect of income effects. Changing levels of $\bar{V}_t$ will require different compensation schemes. Hence, the exact location of λ^{***} depends on the particular compensation scheme as the associated income effects may slightly change preferences for λ, i.e. shift the marginal cost and benefit curves displayed in Diagram 2.

3 Extensions

So far attention was confined to an economy where the safe rate-of-return is constant over time. The prospects of a third debt instrument, namely a defined benefit social security system, have also been neglected. In a first step, we now show that a time-varying, safe rate-of-return does not alter the quality of the foregoing conclusions and that defined benefits are equivalent to safe bonds. Finally a second group of representative agents who do not invest in the stock market (risky technology) is introduced into our model. In this setting we show that both groups require different social security contribution rates, i.e. debt structures. If either is at a corner solution there is additional scope for an intragenerational reallocation of the public debt.

3.1 Time-Varying Safe Returns

To work out the pivotal elements, the safe rate of return was assumed to remain constant over time. However, the main results of our previous analysis carry over to an economy where r is now an i.i.d. random variable. Regarding generation 0, nothing is changed, i.e. the agents and the social planner start maximizing after r_1 is known. Except for the additional expectations regarding r the long-run planning problem (19) is also little changed:

$$\begin{aligned}\max_{\lambda} V_t = \quad & E_{w_t,r_{t+1}}\Big[U(w_t(1-\lambda\frac{g_0}{w_0(1+n)}) - \frac{r_{t+1}-n}{(1+r_{t+1})}\frac{(1-\lambda)g_0}{(1+n)} - a_t - h_t)\Big] \\ & +\beta E_{w_t w_{t+1} R_{t+1} r_{t+1}}\Big[U(a_t(1+r_{t+1}) + h_t(1+R) + \lambda\frac{g_0}{w_0}w_{t+1})\Big].\end{aligned}$$

Employing the envelope conditions (13) and (14), yields:

$$\begin{aligned}\frac{dV_t}{d\lambda} = \quad & \frac{g_0}{(1+n)}\Big(E_{wr}\Big[\frac{r_{t+1}-n}{1+r_{t+1}}U'(c^1)\Big]\frac{w_0-E[w]}{w_0} \\ & -cov_{w_t r_{t+1}}(U'(c^1),\frac{w_t}{w_0}) + \beta(1+n)cov_{r_{t+1}w_t w_{t+1}R}(U'(c^2),\frac{w_{t+1}}{w_0})]\Big) = 0.\end{aligned} \tag{27}$$

Due to the nature of the treasury's tax schedule (11), the initial interest rate r_1 does not, unlike the wage rate w_0, enter into the long-run first order condition. While there are now additional expectations regarding the safe rate-of-return, the principal structure of the first order condition is preserved. Regarding our Pareto-improving interventions that were discussed in Section 2.5, we note that the government can still reallocate gains and losses along its budget constraint. However, each compensation scheme will now require some sort of risk-taking.

3.2 Defined Benefits

We will now briefly show that a defined benefit system is equivalent to an explicit debt scheme. The budget constraint of a defined benefit system, which is used to roll over a fraction γ of the public debt, is given by:

$$\tau_t^{DB} w_t = \frac{\gamma g_0}{(1+n)}, \quad p_t^{DB} = \gamma g_0. \tag{28}$$

Once we recall that the young agent can consume c^1, invest an amount a into safe assets and an amount h into risky assets, the present value budget constraint is given by:

$$c_t^1 + a_t + h_t = w_t(1 - \tau_t^{DB} - \tau_t^t) + \frac{p_{t+1}^{DB}}{(1+r_{t+1})}. \tag{29}$$

Utilizing (28) and (11) where $(1-\lambda)$ is replaced by $(1-\gamma)$, the right-hand side of (29) can now be rewritten such that:

$$c_t^1 + a_t + h_t = w_t - \frac{g_0(r_{t+1}-n)}{(1+n)(1+r_{t+1})}. \tag{30}$$

Hence the structure of debt γ is irrelevant, i.e. a defined benefit system is equivalent to a bond-financed debt scheme.

3.3 A Working Class

This final paragraph considers a society that is partitioned into a group of capitalists who are endowed with a large amount of efficient labor and a group of workers with a low labor endowment. While capitalists participate in the stock-market, workers invest in the safe technology only.[19,20] The working class is assumed to make up a fraction α of the population and each worker has only a fraction ϕ of the effective labor endowment of a capitalist. Hence, workers earn a fraction $\theta = \frac{\alpha\phi}{1+\alpha(\phi-1)}$ of aggregate wages. Consequently, with a linear social security tax, the debt rolled over on the shoulders of workers and capitalists is given by $g_0^w = \theta\frac{g_0}{1+n}$ and $g_0^c = (1-\theta)\frac{g_0}{1+n}$. Workers will now choose safe investment according to (13). The optimal shares of debt for the working class, $\lambda_w^*, \lambda_w^{**}$ are then characterized by (17) and (19), with the notable difference that $h = 0$.[21] For $Ew = w_0$, we therefore have $\frac{dV_0^w}{d\lambda_w}_{|\lambda_w=0} = 0$ and $\frac{dV_t^w}{d\lambda_w}_{|\lambda_w=0} > 0$ and $\frac{d^2V_t^w}{(d\lambda_w)^2} < 0$; i.e. a unique globally optimal debt structure λ_w^{***} exists if g_0 is large enough (see (39) in Appendix 5.2). If per capita debt $\frac{g_0}{1+n}\theta$ is not large enough to transport a sufficient amount of wage-related risk into the retirement period, we have $\lambda_w^{***} > 1$ and hence, $\frac{dV^w}{d\lambda_w}_{|\lambda_w=1} > 0$. Once $\lambda_c^{***} < 1$, bonds from the capitalists' debt scheme can be injected into the workers' pension scheme. If the capitalists, in turn, pay the implicit tax associated with this debt swap as a subsidy to the workers, the marginal increase in rent for workers is, recalling equations (21)-(25) with $\lambda_w = 1$, given by:

$$\frac{\partial \mathcal{L}^w}{\partial g^w} = \frac{1}{g^w}\frac{\partial V_0}{\partial g^w}\left(\frac{\frac{\partial V_0^w}{\partial \lambda^w}}{\frac{\partial V_0^w}{\partial g^w}} - \frac{\frac{\partial V_t^w}{\partial \lambda^w}}{\frac{\partial V_t^w}{\partial g^w}}\right) > 0. \tag{31}$$

Thus, while utility of the capitalists remains constant, the utility of workers has increased.

To a certain extent this result illustrates the main point of our analysis. Given that we already have incurred the debt, the risk sharing capacities of the debt are a scarce resource. Transferring some of the debt from capitalists to workers improves risk sharing without any additional crowding-out of capital.

4 Conclusion

If a government can issue safe bonds and claims to an unfunded social security system to service a given obligation, there exists a set of Pareto-efficient financing policies. This set is characterized by the conflicting interests of agents who are currently alive and those who are yet unborn. The current young, who have already observed their wage income, will prefer safe debt, i.e. safe retirement benefits. The unborn generations on the contrary

[19] At this point, we take the non-participation of workers in the stock-market as given; Abel (2001) endogenizes the participation decision by introducing fixed costs that make it rational for agents with a small portfolio to abstain from the stock market. Regarding this non-participation decision, Diamond and Geanakoplos (2003) point out that roughly 50 percent of the working population in the US does not hold any stocks (this figure includes indirect holdings of stocks through pension plans).

[20] To focus on the intertemporal and intergenerational reallocation of risks, rather than intragenerational redistribution which can also be achieved without social security, we assume that the affiliation with the two groups of all agents is known in period $t = 0$, i.e. cannot be insured against.

[21] Given the different labor endowment and the different exposition to the covariance risk ($cov(R, w_{t+1})$), it is clear that it is not optimal to choose a "one-size-fits-all pension scheme". Hence we will right away allow for distinct debt structures λ_c, λ_w for capitalists and workers.

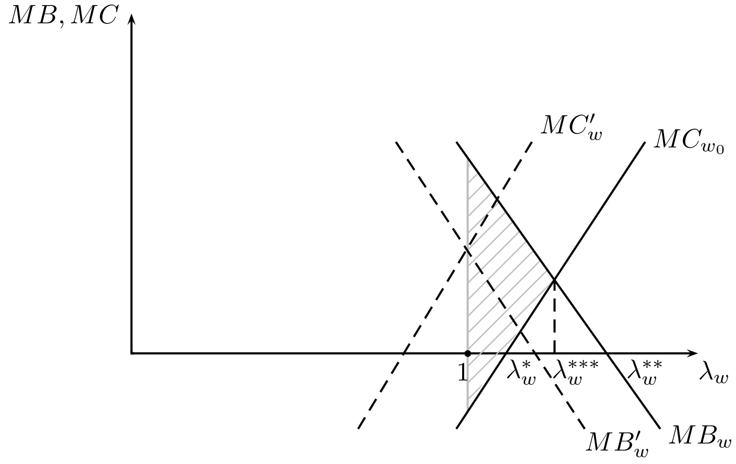

Diagram 4: *Intragenerational reallocation of the debt.*

The shaded area to the right of $\lambda = 1$ *is the welfare gain associated with an intragenerational debt swap.*

benefit from the ex-ante diversification of their wage risk if a large portion of the initial debt is injected into social security.

The government may now act as a representative of the unborn members of society. Through its budget constraint, it can offer generation 0 a compensation that reflects the willingness to pay of all unborn agents. Such an intermediation allows to collect the benefits, which are associated with the optimum structure for government debt λ^{***} in a Pareto-improving manner. If the initial conditions are such that $\lambda^{***} > 0$, an unfunded social security system is therefore always warranted.

Unlike the deterministic economy, where all debt policies are equally desirable, the current analysis shows that the structure of government debt has distinct implications for individual welfare. If we compare our analysis to the problem of optimal capital accumulation, the following analogy is notable: While the golden rule capital intensity maximizes long-run utility, it comes at the cost of lower consumption along the transition path. All capital intensities below the golden rule level are therefore efficient and there is no compensation mechanism available.[22] Compared to the reallocation of aggregate risks, the situation without compensation is similar; there exists a whole set of efficient debt structures. In the present case, however, the government budget constraint *can* be used to reconcile the conflicting interests of the current young and those who live in the long run in a Pareto-improving manner. As a result, subject to our assumptions, the set of efficient debt structures can be narrowed.

[22]The lack of such a compensation mechanism led to the turnpike literature; see e.g. Samuelson (1968) or Blanchard and Fischer (1989). The absence of such an intertemporal compensation mechanism is of course also the reason for the intertemporal efficiency of pay-go schemes that we have been referring to in Footnote 2.

5 Appendix

5.1 The Envelope Conditions

Derivation of condition (19): Equations (13) and (14) imply an investment behavior for each realization of the wage-income w_t, namely $a_t = a_t(w_t, \lambda)$, $h_t = h_t(w_t, \lambda)$. Hence, agents smooth consumption state by state with regard to first period wage income. At the same time, they smooth consumption in expectations when it comes to second period consumption. Taking expectations E_{w_t} of (13) and (14) yields:

$$E_{w_t}[U'(c^1)] = \beta(1+r_1)E_{w_t}\Big[E_{w_{t+1}R}[U'(c^2)]\Big], \tag{32}$$

$$E_{w_t}[U'(c^1)] = \beta E_{w_t}\Big[E_{w_{t+1}R}[(1+R)U'(c^2)]\Big]. \tag{33}$$

Writing out the first order condition for λ^{**}, we obtain:

$$\begin{aligned}\frac{dV_t}{d\lambda} = &\Big(E_{w_t}\Big[-U'(c^1)w_t + (1+n)\beta E_{w_{t+1}R}[w_{t+1}U'(c^2)]\Big]\frac{g_0}{w_0} \\ &+ \frac{r-n}{(1+r)}g_0 E_{w_t}\Big[U'(c^1)\Big]\Big)\frac{1}{1+n} \\ &- E_{w_t}[U'(c^1)(\frac{da}{d\lambda}+\frac{dh}{d\lambda}) - \beta E_{w_{t+1}R}[U'(c^2)((1+r)\frac{da}{d\lambda}+(1+R)\frac{dh}{d\lambda})]] = 0.\end{aligned} \tag{34}$$

To rearrange the first line in (34), equation (32) can be utilized as $\frac{E_{w_t}[U'(c^1)]}{1+r} = \beta E_{w_t}[U'(c^2)]$. Applying the covariance rule ($E[xy] = cov(x,y) + E[x]E[y]$) to the resulting expressions, we obtain (19). Noting that the derivatives $\frac{da}{d\lambda}$ and $\frac{dh}{d\lambda}$ are functions of w_t, the second line can be rearranged using the covariance rule such that:

$$\begin{aligned}&-E_{w_t}[U'(c^1)]E_{w_t}[\frac{da}{d\lambda}] + (1+r)\beta E_{w_t w_{t+1}R}[U'(c^2)]E_{w_t}[\frac{da}{d\lambda}] \\ &-E_{w_t}[U'(c^1)]E_{w_t}[\frac{dh}{d\lambda}] + \beta E_{w_t w_{t+1}R}[(1+R)U'(c^2)]E_{w_t}[\frac{da}{d\lambda}] \\ &+cov_{w_t}(-U'(c^1) + (1+r)\beta E_{w_{t+1}R}[U'(c^2)], \frac{da}{d\lambda}) \\ &+cov_{w_t}(-U'(c^1) + \beta E_{w_{t+1}R}[(1+R)U'(c^2)], \frac{dh}{d\lambda}) = 0.\end{aligned}$$

That is, recalling (13), (14), (32), and (33), the expressions related to changes in the investment behavior vanish by the envelope theorem.

5.2 Characteristics of the Long-run Optimum

This appendix examines the properties of condition (19). In a first step we note that (19) characterizes a "best" debt structure, which may or may not be interior. In a next step it is shown that interior solutions will exist for appropriate parameters. Finally the conditions, which ensure that $\frac{dV_t(\lambda)}{d\lambda}_{|\lambda=0} > 0$ and that $\frac{d^2V_t(\lambda)}{d\lambda^2} < 0$, are outlined.

1) Existence Since short sales of bonds or social security claims were ruled out, the set of feasible debt structures $[0, 1]$ is a compact subset of $\mathbb{R}$. If $V_t(\lambda)$ is continuous and real-valued, it will therefore attain its bounds on this choice set according to the Weierstrass theorem.

2) Interior Solutions If $\frac{dcov(U'(c^1),w_t)}{d\tau^s}\frac{d\tau^s}{d\lambda}$ and $\frac{dcov(U'(c^2),w_{t+1})}{d\tau^s}\frac{d\tau^s}{d\lambda}$ are continuous and $\frac{dh}{d\tau^s} < 0$, it is obvious that for sufficiently large g_0, sufficiently small $cov(R, w_{t+1})$, and $E_w[w] = w_0$ or $r = n$, we have:

$$\frac{dV_t}{d\lambda}_{|\lambda=0} > 0, \quad \frac{dV_t}{d\lambda}_{|\lambda=1} < 0. \tag{35}$$

In this case, there exists one interior global optimum λ^{**} and there may exist several local optima.

3) Unique Optimum To interpret condition (19) in more detail, we will first show that $cov(U'(c^1), w_t) < 0$ and give a condition for $cov(U'(c^2), w_{t+1}) \gtreqless 0$:

$$\begin{aligned} cov(c^1, w_t) = & \quad cov((1-\tau^s)w_t - \frac{r-n}{1+r}\frac{g_0}{1+n}(1-\lambda) - s(w_t, \tau^s), w_t) \\ = & \quad cov((1-\tau^s)w_t - s(w_t, \tau^s), w_t) > 0, \end{aligned} \tag{36}$$

where the sign $cov(c^1, w_t) > 0$ is due to the normality of c^1; i.e. $\frac{\partial((1-\tau^s)w_t - s(w_t,\tau^s))}{\partial w_t} > 0$. Hence, since $U''() < 0$, $cov(U'(c^1), w_t) < 0$. For $cov(U'(c^2), w_{t+1})$ we have:

$$\begin{aligned} cov(c^2, w_{t+1}) = & \quad cov((1+r)a + (1+R)h + \tau^s(1+n)w_{t+1}, w_{t+1}) \\ = & \quad hcov(R, w_{t+1}) + \tau^s(1+n)\sigma_w^2 \gtreqless 0; \quad \tau^s = \lambda\frac{g_0}{(1+n)w_0}. \end{aligned} \tag{37}$$

Hence, depending on the amount of risky assets h, $cov(w, R) \gtreqless 0$ and the amount of debt that is injected in the pension system, we may have $cov(U'(c^2), w_{t+1}) \gtreqless 0$. Together with the ambiguous sign of $\frac{(n-r)(Ew-w_0)}{w_0(1+r)}$, we may or may not have $\frac{dV_t}{d\lambda}_{|\lambda=0} > 0$.

4) Sufficient Condition To allow for a global optimum, it is a sufficient condition, that $\frac{dV_t}{d\lambda}$ is downward-sloping in λ:

$$\frac{d^2V_t}{(d\lambda)^2} = \frac{g_0}{1+n}\Big(\frac{n-r}{1+r}\frac{E[w]-w_0}{w_0}\frac{dE[U'(c^1)]}{d\lambda} \tag{38}$$

$$- \frac{dcov_{w_t}(U'(c^1), \frac{w_t}{w_0})}{d\lambda} + \beta(1+n)\frac{dcov_{w_{t+1}R}(U'(c^2), \frac{w_{t+1}}{w_0})}{d\lambda}\Big) < 0.$$

A first inspection of (38) indicates that for $Ew = w_0$ and/or $r = n$, we expect $\frac{dcov(U'(c^1),w_t)}{d\lambda} > 0$, $\frac{dcov(U'(c^2),w_{t+1})}{d\lambda} < 0$ and thus $\frac{d^2V_t}{d\lambda^2} < 0$.[23] With respect to $\frac{dcov(U'(c^1),w_t)}{d\lambda}$ we have:

$$\begin{aligned} \frac{dcov_{w_t}(U'(c^1), w_t)}{d\lambda} &= cov_{w_t}(U''(c^1)(-w - \frac{\partial s}{\partial \tau^s})\frac{d\tau^s}{d\lambda}, w) \\ &\approx E_{w_t}[U''(c^1)]cov_{w_t}(-w - \frac{\partial s}{\partial \tau^s}, w)\frac{d\tau^s}{d\lambda} > 0, \end{aligned} \tag{39}$$

where $\frac{\partial(\frac{\partial s}{\partial w})}{\partial \tau^s} = \frac{\partial(\frac{\partial s}{\partial \tau^s})}{\partial w} > -1$, if preferences are homothetic. Moreover, (39) holds with strict equality if $U'''(c^1) = 0$. Finally, by the same approximation as in (39) we have:

$$\frac{dcov(U'(c^2), w_{t+1})}{d\lambda} \approx E_{wR}[U''(c^2)]\Big(\frac{\partial h}{\partial \tau^s}cov(R, w_{t+1}) + (1+n)\sigma_w^2\Big)\frac{d\tau^s}{d\lambda} \gtreqless 0, \tag{40}$$

where (40) is negative if $\frac{\partial h}{\partial \tau}cov(R, w_{t+1}) + (1+n)\sigma_w^2 > 0$. If $cov(R, w_{t+1})$ is large and positive and the share of savings invested in the risky technology is also very large, the crowding-out effect (with regard to risky investment) of additional pension claims may in principle overcompensate the direct effect of the exposition to additional wage-related risks once λ is increased.

References

Aaron, H. (1966). The social insurance paradox. *Canadian Journal of Economics and Political Science*, 32(3):371–374.

Abel, A. B. (1989). The implications of insurance for the efficacy of fiscal policy. *The Journal of Risk and Insurance*, 55(9):339–378.

Abel, A. B. (2001). The effects of investing social security funds in the stock market when fixed costs prevent some households from holding stocks. *American Economic Review*, 91(1):128–148.

Arrow, K. J. (1970). *Essays in the Theory of Risk Bearing.* Amsterdam: North-Holland Publishing.

Ball, L. and Mankiw, G. (2001). Intergenerational risk sharing in the spirit of Arrow, Debreu, and Rawls, with applications to social security design. *NBER Working Paper*, (8270):1–40.

[23]Regarding the first element, which is inherently ambiguous, we note that for $U'''(c^1) > 0$, $\frac{dE[U(c^1)]}{d\lambda}$ is most likely negative, as the variance of first period consumption is decreasing in λ. However, at the same time an increase in λ may increase second period variance and if $U'''() > 0$, precautionary savings (see Green (1977) and Kimball (1990) for the coefficient of prudence) will increase $E[U'(c^1)]$.

Ball, L. and Mankiw, G. (2007). Intergenerational risk sharing in the spirit of arrow, debreu, and rawls, with applications to social security design. *Journal of Political Economy*, 115(4):523–547.

Belan, P. and Pestieau, P. (1999). Privatizing social security: A critical assessment. *The Geneva Papers on Risk and Insurance*, 24(1):114–130.

Bester, H. (1984). Increasing risk and equilibrium under uncertainty. *Journal of Economic Theory*, 33:378–386.

Blanchard, O. J. and Fischer, S. (1989). *Lectures on Macroeconomics*. Cambridge, MA: MIT Press.

Bohn, H. (1998). Risk sharing in a stochastic overlapping generations economy. *Economics Working Paper Series*, (1076):1–44.

Bohn, H. (2003). Intergenerational risk sharing and fiscal policy. *Working paper, University of California at Santa Barbara*, pages 1–35.

Breyer, F. (1989). On the intergenerational pareto efficiency of pay-as-you-go financed pension systems. *Journal of Institutional and Theoretical Economics*, (145):643–658.

Cass, D. and Yaari, M. E. (1966). A re-examination of the pure consumption loans model. *Journal of Political Economy*, 74(4):353–367.

De La Croix, D. and Michel, P. (2002). *A Theory of Economic Growth*. Cambridge: Cambridge Univ. Press.

Diamond, P. A. (1965). National debt in a neoclassical growth model. *American Economic Review*, 55(5):1126–1150.

Diamond, P. A. (1977). A framework for social security analysis. *Journal of Public Economics*, 8(3):275–298.

Diamond, P. A. (1996). Proposals to restructure social security. *The Journal of Economic Perspectives*, 10(3):67–88.

Diamond, P. A. (2000). Towards an optimal social security design. *CeRP Working Paper*, (4):4–17.

Diamond, P. A. and Geanakoplos, J. (2003). Social security investment in equities. *American Economic Review*, 93(4):1047–1074.

Enders, W. and Lapan, H. E. (1982). Social security taxation and intergenerational risk sharing. *International Economic Review*, 23(3):647–658.

Fenge, R. (1995). Pareto-efficiency of the pay-as-you-go pension system with intragenerational fairness. *Finanzarchiv*, 52(3):357–363.

Finetti, B. D. (1952). Sulla preferibilita. *Annali di Economica*, (11):685–709.

Fischer, S. (1983). Welfare aspects of government issue of indexed bonds. In *Inflation, Debt and Indexation*, pages 223–246. ed. R. Dornbusch and M. Simonsen, MIT Press.

Friedman, M. (1999). Speaking the truth about social security reform. *Cato Institute Briefing Papers*, (46):1–3.

Gale, D. (1990). The efficient design of public debt. In *Public debt management: theory and history*, pages 14–41. ed. R. Dornbusch and M. Draghi, Camebridge University Press.

Gordon, R. H. and Varian, H. R. (1988). Intergenerational risk sharing. *Journal of Public Economics*, 37(1):185–202.

Gottardi, P. and Kubler, F. (2008). Social security and risk sharing. *Working Paper*, pages 1–40.

Green, J. R. (1977). Mitigating demographic risk through social insurance. *NBER Working Paper*, (215):1–32.

Kimball, M. S. (1990). Precautionary savings in the small and in the large. *Econometrica*, 58(1):53–73.

Krüger, D. and Kubler, F. (2005). Pareto improving social security reform when financial markets are incomplete. *American Economic Review*, 96(3):737–755.

Lerner, A. P. (1959). Consumption-loan interest and money. *Journal of Political Economy*, 67(5):512–518.

Ludwig, A. and Reiter, M. (2009). Sharing demographic risk - who is afraid of the baby bust. *Working Paper Köln University*, pages 1–47.

Merton, R. (1983). On the role of social security as a means for efficient risk sharing in an economy where human capital is not tradable. In *Financial Aspects of the United States Pension System*, pages 259–290. ed. Z. Bodie and J. Shoven, University of Chicago Press, Chicago.

Pratt, J. W. (1964). Risk aversion in the small and in the large. *Econometrica*, 32(1):122–136.

Samuelson, P. A. (1954). The pure theory of public expenditure. *Review of Economics and Statistics*, 36(4):386–389.

Samuelson, P. A. (1958). An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy*, 66(6):467–482.

Samuelson, P. A. (1959). Consumption-loan interest and money: Reply. *Journal of Political Economy*, 67(5):518–522.

Samuelson, P. A. (1968). The two-part golden rule deduced as the asymptotic turnpike of catenary motions. *Western Economic Journal*, (VI , March, 1968):85–89.

Samuelson, P. A. (1975). Optimum social security in a life-cycle growth model. *International Economic Review*, 16(3):539–544.

Shiller, R. (1999). Social security and institutions for intergenerational, intragenerational and international risk sharing. *Carnegie-Rochester Conference Series on Public Policy*, 50:165–204.

Sinn, H. W. (2000). Why a funded pension system is useful and why it is not useful. *International Tax and Public Finance*, 7(4-5):389–410.

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