MAX-PLANCK-INSTITUT FÜR SOZIALRECHT UND SOZIALPOLITIK
MAX PLANCK INSTITUTE FOR SOCIAL LAW AND SOCIAL POLICY

mea Munich Center for the Economics of Aging

Avoiding Root-Finding in the Krusell-Smith Algorithm Simulation

Ivo Bakota

16-2020

MEA DISCUSSION PAPERS

mea – Amalienstr. 33_D-80799 Munich_Phone +49 89 38602-355_Fax +49 89 38602-390_www.mpisoc.mpg.de

Avoiding Root-Finding in the Krusell-Smith Algorithm Simulation

Ivo Bakota

Abstract:

This paper proposes a novel method to compute the simulation part of the Krusell-Smith (1997, 1998) algorithm when the agents can trade in more than one asset (for example, capital and bonds). The Krusell-Smith algorithm is used to solve general equilibrium models with both aggregate and uninsurable idiosyncratic risk and can be used to solve bounded rationality equilibria and to approximate rational expectations equilibria. When applied to solve a model with more than one financial asset, in the simulation, the standard algorithm has to impose equilibria for each additional asset (find the market-clearing price), for each period simulated. This procedure entails root-finding for each period, which is computationally very expensive. I show that it is possible to avoid this root-finding by not imposing the equilibria each period, but instead by simulating the model without market clearing. The method updates the law of motion for asset prices by using Newton-like methods (Broyden's method) on the simulated excess demand, instead of imposing equilibrium for each period and running regressions on the clearing prices. Since the method avoids the root-finding for each time period simulated, it leads to a significant reduction in computation time. In the example model, the proposed version of the algorithm leads to a 32% decrease in computational time, even when measured conservatively. This method could be especially useful in computing asset pricing models (for example, models with risky and safe assets) with both aggregate and uninsurable idiosyncratic risk since methods which use linearization in the neighborhood of the aggregate steady state are considered to be less accurate than global solution methods for these particular types of models.

Zusammenfassung:

Dieses Papier schlägt eine neuartige Methode zur Berechnung des Simulationsteils des Krusell-Smith (1997, 1998) Algorithmus vor, wenn Agenten mit mehr als einem Vermögenswert (z.B. Kapital und Anleihen) handeln können. Der Krusell-Smith-Algorithmus wird zur Lösung allgemeiner Gleichgewichtsmodelle mit sowohl aggregiertem als auch nicht versicherbarem idiosynkratischen Risiko verwendet und kann zur Lösung begrenzter Rationalitätsgleichgewichte und zur Approximation rationaler Erwartungsgleichgewichte verwendet werden. Bei der Anwendung zur Lösung eines Modells mit mehr als einem finanziellen Vermögenswert muss der Standardalgorithmus in der Simulation Gleichgewichte für jeden zusätzlichen Vermögenswert (Ermittlung des Marktausgleichspreises) für jede simulierte Periode auferlegen. Dieses Verfahren erfordert eine rechnerich aufwendige Nullstellenbestimmung für jede Periode. Ich zeige eine Möglichkeit zur Vermeidung der Nullstellenbestimmung auf, indem die Gleichgewichte nicht für jede Periode auferlegt werden, sondern das Modell ohne Markträumung simuliert wird. Die Methode aktualisiert das Bewegungsgesetz für Vermögenspreise, indem sie Newton-ähnliche Methoden (Broyden-Methode) auf die simulierte Überschussnachfrage anwendet, anstatt für jede Periode ein Gleichgewicht aufzuerlegen und Regressionen auf die Markträumungspreise durchzuführen. Da die Methode die Nullstellenbestimmung für jede simulierte Zeitperiode vermeidet, führt sie zu einer erheblichen Reduzierung der Berechnungszeit. Im Beispielmodell führt die vorgeschlagene Version des Algorithmus selbst bei konservativer Messung zu einer Verringerung der Rechenzeit um 32%. Diese Methode könnte besonders nützlich bei der Berechnung von Preisfindungsmodellen für Vermögenswerte (z.B. Modelle mit riskanten und sicheren Vermögenswerten) mit sowohl aggregiertem als auch nicht versicherbarem idiosynkratischen Risiko sein, da Methoden, die eine Linearisierung in der Nachbarschaft des aggregierten stationären Zustands verwenden, als weniger genau angesehen werden als globale Lösungsmethoden für diese speziellen Modelltypen.

Keywords:

Portfolio Choice, Heterogeneous agents, Krusell-Smith

JEL Classification:

E44, G12, C63

Avoiding Root-Finding in the Krusell-Smith Algorithm Simulation

Ivo Bakota*

Max Planck Institute for Social Law and Social Policy, Munich

October 2, 2020

Abstract

This paper proposes a novel method to compute the simulation part of the Krusell-Smith (1997, 1998) algorithm when the agents can trade in more than one asset (for example, capital and bonds). The Krusell-Smith algorithm is used to solve general equilibrium models with both aggregate and uninsurable idiosyncratic risk and can be used to solve bounded rationality equilibria and to approximate rational expectations equilibria. When applied to solve a model with more than one financial asset, in the simulation, the standard algorithm has to impose equilibria for each additional asset (find the market-clearing price), for each period simulated. This procedure entails root-finding for each period, which is computationally very expensive. I show that it is possible to avoid this root-finding by not imposing the equilibria each period, but instead by simulating the model without market clearing. The method updates the law of motion for asset prices by using Newton-like methods (Broyden's method) on the simulated excess demand, instead of imposing equilibrium for each period and running regressions on the clearing prices. Since the method avoids the root-finding for each time period simulated, it leads to a significant reduction in computation time. In the example model, the proposed version of the algorithm leads to a 32% decrease in computational time, even when measured conservatively. This method could be especially useful in computing asset pricing models (for example, models with risky and safe assets) with both aggregate and uninsurable idiosyncratic risk since methods which use linearization in the neighborhood of the aggregate steady state are considered to be less accurate than global solution methods for these particular types of models.

JEL Classification: E44, G12, C63

Keywords: Portfolio Choice, Heterogeneous agents, Krusell-Smith

*Email: bakota@mea.mpisoc.mpg.de.

I am grateful to Sven Hartjenstein, Marios Karabarbounis, Dirk Krueger, Felix Kübler and Christian Proaño for insightful discussions and useful comments.

1 Introduction

This paper proposes a novel method to compute the simulation part of the Krusell-Smith algorithm when agents can trade with more than one asset. The classic example is the macroeconomic model with both idiosyncratic and aggregate risk, with a borrowing constraint, where agents can choose to save in both risky capital and safe bonds. The idea is to avoid root-finding in the simulation part of the algorithm, where it is necessary to find a market-clearing bond price. Instead, the proposed algorithm lets the economy proceed to the next period with the markets uncleared and updates the perceived law of motions for the bond price based on the simulated excess demand for bonds. The idea of finding a market-clearing price by having information on excess demand can be traced far back in the history of economics, not necessarily as a solution method, but as an actual process by which general equilibrium emerges in existing markets. The process was called tâtonnement (French for "trial and error" or "groping") by Walras (1874) (translated to English: Walras (1954)). The proposed algorithm, however, does not imply anything about the process of reaching equilibrium, but uses the idea purely as a part of the solution algorithm.

The computational gain of using the proposed algorithm is a shorter time duration due to the avoidance of bond market clearing. Market clearing involves a root-finding process, which is computationally very expensive. The root-finding consists of finding a bond interest rate (or equity premium), which will clear the bond market in each simulated period. In the general equilibrium, all the markets are supposed to clear, but in the process of finding the general equilibrium laws of motion, it can be computationally beneficial not to impose market clearing, and use the information on excess demand to make subsequent updates.

The proposed method could be especially useful in computing asset pricing models (for example models with risky and safe assets) with both aggregate and uninsurable idiosyncratic risk, since methods that use linearization in the neighborhood of the aggregate steady state are considered less accurate than global solution methods for these particular types of models. For example, Reiter (2009) proposes a solution using projection and perturbation instead of attempting to represent the cross-sectional distribution of wealth by a small number of statistics in order to reduce the dimensions in state space as in Krusell and Smith (1997), Den Haan

(1997) and Reiter (2002). However, a solution method based on projection and perturbation most likely is not accurate enough for solving the models with asset pricing, as it assumes the linearity in the aggregate variables, which is not sufficient for the problems of portfolio choice and asset pricing (Reiter, 2009). In addition to this specific application, further use of this method could be useful to accelerate the Krusell-Smith algorithm where any type of market-clearing has to be imposed during the simulation of the model (for example clearing of the labor market in a model where labor supply is determined endogenously).

The rest of the paper is organized as follows: Section 2 describes the sample model, Section 3 describes the classical Krusell-Smith algorithm (Krusell and Smith, 1997) used to solve the models with both aggregate and idiosyncratic risk and a portfolio choice, Section 4 illustrates the proposed algorithm, Section 5 shows the computational performance comparisons between the classic and the proposed algorithm. Section 6 discusses the results and potential applications of the proposed algorithm, and Section 7 concludes.

2 Example model

The presented model is based on Algan et al. (2009), and in the tradition of Krusell and Smith (1997). The model consists of a continuum of heterogeneous agents facing aggregate risk, uninsurable idiosyncratic labor risk and a borrowing constraint, and who save in two assets: risky equity and safe bonds. Unlike Algan et al. (2009), the model parsimoniously captures the life cycle dynamics of the households, in the fashion of Krueger et al. (2016), where working-age agents face the retirement shock and retired households face the risk of dying.

2.1 Production technology

In each period t, the representative firm uses aggregate capital K_t, and aggregate labor L_t, to produce y units of final good with the aggregate technology $y_t = f(z_t, K_t, L_t)$, where z_t is an aggregate total factor productivity (TFP) shock. I assume that z_t follows a stationary Markov process with transition function $\Pi_t(z, z') = Pr(z_{t+1} = z'|z_t = z)$. The production function is continuously differentiable, strictly increasing, strictly concave and homogeneous of degree one in K and L. Capital depreciates at the constant rate $\delta \in (0,1)$ and it accumulates according to the standard law of motion:

$$K_{t+1} = I_t + (1-\delta)K_t$$

where I_t is aggregate investment. The particular aggregate production technology is:

$$Y_t = z_t A K_t^{\Delta} L_t^{1-\Delta}$$

2.2 Parsimonious life-cycle structure

In each period, working-age households have a chance of retiring θ, and retired households have a chance of dying v, similarly as in Castaneda et al. (2003) and Krueger et al. (2016). Therefore the share of working age households in the total population is:

$$\Pi_W = \frac{1-v}{(1-\theta)+(1-v)}$$

and the share of the retired households in the total population is:

$$\Pi_R = \frac{1-\theta}{(1-\theta)+(1-v)}$$

The retired households who die in period t are replaced by new-born agents who start at a working age without any assets. For simplicity, the retired households have perfect annuity markets, which make their returns larger by a fraction of $\frac{1}{v}$, as in Krueger et al. (2016). This life-cycle structure with stochastic aging and death helps capture important life-cycle aspects of the economy and risks that households face without adding an excessive computational burden.

2.3 Preferences

Households are indexed by i, and they have Esptein-Zin preferences (Epstein and Zin, 1989). These preferences are often used in asset-pricing models, since they allow one to separate the intertemporal elasticity of substitution and risk aversion.

Households are maximizing their lifetime utility, expressed recursively for the retired agents:

$$V_{R,i,t} = \{c_t^{1-\rho} + \upsilon\beta[E_t V_{R,i,t+1}^{(1-\alpha)}]^{\frac{1-\rho}{1-\alpha}}\}^{\frac{1}{1-\rho}}$$

where $V_{R,i,t}$ is the recursively defined value function of a retired household i, at time period t.

Working-age agents maximize:

$$V_{W,i,t} = \{c_t^{1-\rho} + \beta\big[(1-\theta)E_t V_{W,i,t+1}^{1-\alpha} + \theta E_t V_{R,i,t+1}^{1-\alpha}\big]^{\frac{1-\rho}{1-\alpha}}\}^{\frac{1}{1-\rho}}$$

where $V_{i,t}$ is recursively defined value function of household i, at time period t. Furthermore, β denotes the subjective discount factor, E_t denotes expectations conditional on information at time t, α is the risk aversion, $\frac{1}{\rho}$ is the intertemporal elasticity of substitution.

2.4 Idiosyncratic uncertainty

In each period, working-age households are subject to idiosyncratic labor income risk that can be decomposed into two parts. The first part is the employment probability that depends on aggregate risk and is denoted by $e_t \in (0,1)$. $e = 1$ denotes that the agent is employed, and $e = 0$ that the agent is unemployed. Conditional on z_t and z_{t+1}, I assume that the period $t+1$ realization of the employment shock follows the Markov process.

$$\Pi_e(z, z', e, e') = Pr(e_{t+1} = e'|e_t = e, z_t = z, z_{t+1} = z')$$

This labor risk structure allows idiosyncratic shocks to be correlated with the aggregate productivity shocks, which is consistent with the data and generates the portfolio choice profile such that the share of wealth invested in risky asset is increasing in wealth. The condition imposed on the transition matrix and the law of large numbers implies that the aggregate employment is only a function of the aggregate productivity shock.

If $e = 1$ and the agent is employed, one can assume that the agent is endowed with $l_t \in L \equiv \{l_1, l_2, l_3, ...l_m\}$ efficiency labor units, which she can supply to the firm. Labor efficiency is independent of the aggregate productivity shock, and is governed by a stationary Markov process with transition function $\Pi_l(l, l') = Pr(l_{t+1} = l'|l_t = l)$. If the agent is unemployed, (s)he receives unemployment benefits $g_{u,t}$, which are financed by the government.

2.5 The representative firm

As in Algan et al. (2009), firm leverage in this model is given exogenously. The leverage of the firm is determined exogenously, by the parameter λ. The Modigliani-Miller theorem (1958, 1963) does not hold, as some of the agents are borrowing constrained, and some are portfolio constrained. Therefore, theoretically, the leverage of the firm has some macroeconomic relevance. Additionally, debt is taxed differently than equity returns, and this additionally breaks the Modigliani-Miller theorem.

In the economy, the representative firm can finance its investment with two types of contracts. The first is a one-period risk-free bond, that promises to pay a fixed return to the owner. The second is risky equity that entitles the owner to claim the residual profits of the firm after the firm pays out wages and debt from the previous period. Both of these assets are freely traded in competitive financial markets. By construction, there is no default in the equilibrium.

The return on the bond r^b_{t+1} is determined by the clearing of the bond market:

$$\int_S g^{b,j,e} d\mu = \lambda K'$$

where $g^{b,j,e}$ are the individual policy functions for bonds.

In each period t, the firm redistributes all the residual value of the firm, after production and depreciation have taken place, and wages and debt has been paid. Therefore, the return on the risky equity depends on the realizations of the aggregate shocks and is given by the following equation:

$$(1 + r^s_{t+1}) = \frac{f(z_{t+1}, K_{t+1}, L_{t+1}) - f_L(z_{t+1}, K_{t+1}, L_{t+1})L_{t+1} - \lambda K_{t+1}(1 + r^b_{t+1}) + (1 - \delta)K_{t+1}}{(1 - \lambda)K_{t+1}}$$

An important caveat in having heterogeneous households that own the firm is that they do not necessarily have the same stochastic discount factor m^j_{t+1}, and therefore the definition of the objective function of the firm is not straightforward. I follow Algan et al. (2009), who assume that the firm is maximizing the welfare of the agents who have interior portfolio choice, and consequently the firm has the same stochastic discount factor m_{t+1} as the agents with the interior portfolio choice.

As in Algan et al. (2009), it is possible use the fact that for a given stochastic discount factor $V_t = K_{t+1}$, which enables the elimination of the capital Euler equation from the equilibrium conditions.

2.6 Financial markets

As stated earlier, households can save in two assets: risky equity and safe bonds (firm debt). There are borrowing constraints for both assets, the lowest amounts of equity and debt that households can hold in period t are: κ^s and κ^b, respectively. Markets are assumed to be incomplete in the sense that there are no markets for the assets contingent on the realization of individual idiosyncratic shocks. Furthermore, if the household wants to save a positive amount of resources in equity in the period t, it has to pay ϕ as a per period cost of participating in the stock market.

2.7 Government

The government runs a unemployment insurance program, which is modeled as in Krueger et al. (2016) and is financed by special labor income taxes. Unemployment benefits are financed with a labor tax rate τ^u_t. The amount of the unemployment benefits $g_{u,t}$ is determined by a constant η, which represents the fraction of the average wage in each period.

To satisfy the budget constraint, government has to tax labor at the tax rate:

$$\tau^u_t = \frac{1}{1 + \frac{1-\Pi_u(z)}{\Pi_u(z)\eta}}$$

where Π_u is the share of unemployed people in the total working age population.

2.8 Household problem

Household i maximizes its expected lifetime utility subject to the constraints below:

$$c_{i,t} + s_{i,t+1} + b_{i,t+1} + \phi \mathbb{I}_{\{s_{i,t+1} \neq 0\}} \leq \omega_{i,t}$$

$$\omega_{i,t+1} = \begin{cases} w_{t+1} l_{i,t+1}(1 - \tau^l_{t+1}) + (1 + r^s_{t+1}) s_{i,t+1} + (1 + r^b_{t+1}) b_{i,t+1} & \text{if } e = 1 \\ g_{u,t+1}(1 - \tau^l_{t+1}) + (1 + r^s_{t+1}) s_{i,t+1} + (1 + r^b_{t+1}) b_{i,t+1} & \text{if } e = 0 \end{cases}$$

$$(c_{i,t}, b_{i,t+1}, s_{i,t+1}) \geq \left(0, \kappa^b, \kappa^s\right)$$

2.9 Recursive household problem

The idiosyncratic state variables of the household problem are: current wealth ω, current employment and productivity state e, l. Θ denotes the vector of all discrete individual states (all except the current wealth).[1]

The aggregate state variables of the household problem are: state of the TFP shock: z, and distribution which is captured by the probability measure μ. μ is a probability measure on (S, β_s), where $S = [\underline{\omega}, \overline{\omega}] \times \Theta$, and β_s is the Borel σ-algebra. $\underline{\omega}$ and $\overline{\omega}$ denote the minimal and maximal allowed amount of wealth the household can hold.[2] Therefore, for $B \in \beta_s$, $\mu(B)$ indicates the mass of households whose individual states fall in B. Intuitively, one can think of μ as a distribution variable that measures the amount of agents in a certain interval of wealth, for each possible combination of other idiosyncratic variables.

The recursive household problem for the retired households is:

$$v_R(\omega; z, \mu, \delta) = \max_{c,b',s'} \left\{ u(c - \gamma)^{1-\rho} + \upsilon \beta E_{z',\mu',\delta'|z,\mu,\delta} [v_R^m(\omega'; z', \mu', \delta')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

[1] In the benchmark model, there will be 5 elements of Θ: three levels of productivity for the employed households, unemployment, and retirement.

[2] $\underline{\omega}$ is determined by the borrowing constraint, and $\overline{\omega}$ is chosen such that there are always no agents with that amount of wealth in equilibrium.

subject to:

$$c + s' + b' + \phi \mathbb{I}_{\{s' \neq 0\}} = \omega$$

$$\omega' = T'_{ss} + \left[s'(1 + r'^s) + b'(1 + r'^b)\right] \frac{1}{v}$$

$$\mu' = \Gamma(\mu, z, z', d, d')$$

$$(c, b', s') \geq \left(0, \kappa^b, \kappa^s\right)$$

The recursive household problem for the working-age households is:

$$v_W(\omega, e, l; z, \mu, \delta) =$$

$$\max_{c,b',s'} \left\{ u(c-\gamma)^{1-\rho} + \beta E_{e',l',z',\mu',\delta'|e,l,z,\mu,\delta}[(1-\theta)v_W(\omega', e', l'; z', \mu', \delta')^{1-\alpha} + \theta v_R(\omega', e', l'; z', \mu', \delta')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

subject to:

$$c + s' + b' + \phi \mathbb{I}_{\{s' \neq 0\}} = \omega$$

$$\omega' = \begin{cases} w'l'(1-\tau'^l) + s'(1 + r'^s(1-\tau_s)) + b'(1+r'^b) & \text{if } e = 1 \\ g'_u w'l'(1-\tau'^l) + s'(1 + r'^s(1-\tau_s)) + b'(1+r'^b) & \text{if } e = 0 \end{cases}$$

$$\mu' = \Gamma(\mu, z, z', d, d')$$

$$(c, b', s') \geq \left(0, \kappa^b, \kappa^s\right)$$

$$v(\omega; z, \mu) = \max_{c,b',s'} \left\{ u(c-\gamma)^{1-\rho} + \beta E_{z',\mu'|z,\mu}[v(\omega'; z', \mu')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

subject to:

$$c + s' + b' + \phi \mathbb{I}_{\{s' \neq 0\}} = \omega$$

$$\omega' = T'_{ss} + \left[s'(1 + r'^s) + b'(1 + r'^b)\right] \frac{1}{v}$$

where ω is the vector of individual wealth of all agents, μ is the probability measure generated by set $\Omega x E x L$, $\mu' = \Gamma(\mu, z, z')$ is a transition function and $'$ denotes the next period.

2.10 General equilibrium

The economy-wide state is described by $(\omega, e; z, \mu)$. Therefore the individual household policy functions are: $c^j = g^{c,j}(\omega, e, l; z, \mu)$, $b'^j = g^{b,j}(\omega, e, l; z, \mu)$ and $s'^j = g^{s,j}(\omega, e, l; z, \mu)$, and law of motion for the aggregate capital is $K' = g^K(\omega, e, l; z, \mu)$.

A recursive competitive equilibrium is defined by the set of individual policy and value functions $\{v_R, g^{c,R}, g^{s,R}, g^{b,R}, v_W, g^{c,W}, g^{s,W}, g^{b,W}\}$, the laws of motion for the aggregate capital g^K, a set of pricing functions $\{w, R^b, R^s\}$, government policies in period t: $\{\tau^l\}$, and forecasting equations g^L, such that:

1. The law of motion for the aggregate capital g^K and the aggregate "wage function" w, given the taxes satisfy the optimality conditions of the firm.

2. Given $\{w, R^b, R^s\}$, the law of motion Γ, the exogenous transition matrices $\{\Pi_z, Pi_e, Pi_l\}$, the forecasting equation g^L, the law of motion for the aggregate capital g^K, and the tax rates, the policy functions $\{g^{c,j}, g^{b,j}, g^{s,j}\}$ solve the household problem.

3. Labor, shares and the bond markets clear (goods market clears by Walras' law):

   - $$L = \int_S el d\mu$$

   - $$\int_S g^{s,j}(\omega, e, l; z, \mu)\, d\mu = (1 - \lambda)K'$$

   - $$\int_S g^{b,j}(\omega, e, l; z, \mu)\, d\mu = \lambda K'$$

4. The law of motion $\Gamma(\mu, z, z')$ for μ is generated by the optimal policy functions $\{g^c, g^b, g^s\}$, which are endogenous, and by the transition matrices for the aggregate shocks z .[3] Additionally, the forecasting equation for aggregate labor is consistent with the labor market clearing: $g^L(z') = \int_S \epsilon l d\mu$.

5. Government budget constraints are satisfied:

[3] μ' is given by a function Γ, i.e. $\mu' = \Gamma(\mu, z, z', d, d')$

$$T_t^{ss} = \frac{L_t}{\Pi_R} w_t L_t \tau^{lss}$$

$$\tau_t^l = \frac{1}{1 + \frac{1-\Pi_u(z)}{\Pi_u(z)\phi}}$$

3 Classical solution algorithm

1. Guess the law of motion for aggregate capital K_{t+1} and equity premium P_t^e. This means guessing the starting 8 coefficients following the equations (since there are two possible realizations of z):

$$lnK' = a_0(z) + a_1(z)lnK$$

$$lnP^e = b_0(z) + b_1(z)lnK'$$

2. Given the perceived laws of motion, solve the individual problem described earlier. In this step, the endogenous grid method (Carroll, 2006) is used. Instead of constructing the grid on the state variable ω, and and searching for the optimal decision for savings $\tilde{\omega}$, this method creates a grid on the optimal savings amounts $\tilde{\omega}$, and evaluates the individual optimality conditions to obtain the level of wealth ω at which it is optimal to save $\tilde{\omega}$. This way, the root-finding process is avoided, since finding optimal ω given $\tilde{\omega}$, involves only the evaluation of a function (households optimality condition). However, root-finding process is necessary to find the optimal portfolio choice of the household, which is carried out after finding the optimal pairs ω and $\tilde{\omega}$.

3. Simulate the economy, given the perceived aggregate laws of motion. To keep track of wealth distribution, instead of a Monte Carlo simulation, the method proposed by Young (2010) is used. For each realized value of ω, the method distributes the mass of agents between two grid points: ω_i and ω_{i+1}, where $\omega_i < \omega < \omega_{i+1}$, based on the distance of ω, based on Euclidean distance between ω_i, ω and ω_{i+1}. Do this in the following steps:

   (a) Set up an initial distribution in period 1: μ over a simulation grid $i = 1, 2, ...N_{sgrid}$, for each pair of efficiency and employment status, where N_{sgrid} is the number of wealth grid points. Set up an initial value for aggregate states z.

   (b) Find the bond interest rate (expected equity premium P^e) in the given period, which clears the market for bonds. This is performed by iterating on P^e (or on a bond return), until the following equation is satisfied (bond market clears)[4]

$$\sum g^b(\omega, e, l; z, K, P^e)d\mu = \lambda \sum \{g^b(\omega, e, l; z, K, P^e)d\mu + g^s(\omega, e, l; z, K, P^e)d\mu\}$$

where $g^b(\omega, e, l; z, K, P^e)$ and $g^s(\omega, e, l; z, K, P^e)$ are the policy functions for bonds and shares that solve the following recursive household maximization problems: Retired households:

$$v(\omega; z, \mu, P^e) = \max_{c,b',s'} \left\{ u(c-\gamma)^{1-\rho} + \beta E_{z',\mu',P^{e'}|z,\mu,P^e}[v(\omega'; z', \mu', P^{e'})^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

where v is the value function, obtained in step 2. In this step, an additional state variable is included explicitly: expected equity premium P^e.

(c) Depending on the realization for z' , compute the joint distribution of wealth, labor efficiency and employment status.

(d) To generate a long time series of the movement of the economy, repeat substeps b) and c).

4. Use the time series from step 2 and perform a regression of lnK' and P^e on constants and lnK, for all possible values of z and d. This way, the new aggregate laws of motion are obtained.

5. Make a comparison of the laws of motion from step 4 and step 1. If they are almost identical and their predictive power is sufficiently good, the solution algorithm is completed. If not, make a new guess for the laws of motion, based on a linear combination of laws from steps 1 and 4. Then, proceed to step 2.

[4] Similar to Algan et al. (2009), the iteration is performed using the bisection until the excess demand is relatively close to zero, and then the updating is continued using the secant method.

4 Proposed solution algorithm

1. Guess the law of motion for aggregate capital K_{t+1} and equity premium P_t^e. This means guessing all initial coefficients. In this particular case, this would mean 8 coefficients overall, since both relationships are assumed to be linear, and there are two possible realizations of aggregate state z (2 equations $\times$ 2 coefficients $\times$ 2 aggregate states).

$$lnK' = a_0(z) + a_1(z)lnK$$

$$lnP^e = b_0(z) + b_1(z)lnK'$$

2. Given the perceived laws of motion, solve the individual problem described earlier. In this step, the endogenous grid method (Carroll, 2006) is used. Instead of constructing the grid on the state variable ω, and searching for the optimal decision for savings $\tilde{\omega}$, this method creates a grid on the optimal savings amounts $\tilde{\omega}$, and evaluates the individual optimality conditions to obtain the level of wealth ω at which it is optimal to save $\tilde{\omega}$. This way, the root-finding process is avoided, since finding optimal ω, given $\tilde{\omega}$, involves only the evaluation of a function (households optimality condition). However, the root-finding process is necessary to find the optimal portfolio choice of the household, which is performed after finding the optimal pairs ω and $\tilde{\omega}$.

3. Simulate the economy, given the perceived aggregate laws of motion. To keep track of wealth, instead of a Monte Carlo simulation, the method proposed by Young (2010) is used. For each realized value of ω, the method distributes the mass of agents between two grid points: ω_i and ω_{i+1}, where $\omega_i < \omega < \omega_{i+1}$, based on the distance of ω, based on Euclidean distance between ω_i, ω and ω_{i+1}. Do this in the following steps:

   (a) Set up an initial distribution in period 1: μ over a simulation grid $i = 1, 2, ...N_{sgrid}$,for each pair of efficiency and employment status, where N_{sgrid} is the number of wealth grid points. Set up an initial value for aggregate states z.

   (b) Simulate the economy given the perceived laws of motion.

$$\sum g^b(\omega, e, l; z, K)d\mu = \lambda \sum \left\{ g^b(\omega, e, l; z, K)d\mu + g^s(\omega, e, l; z, K)d\mu \right\}$$

where $g^b(\omega, e, l; z, K)$ and $g^s(\omega, e, l; z, K)$ are the policy functions for bonds and shares.

$$v(\omega; z, \mu) = \max_{c,b',s'} \left\{ u(c-\gamma)^{1-\rho} + \beta E_{z',\mu'|z,\mu}[v(\omega'; z', \mu')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

The market for bonds will not necessarily clear. Instead, in each period there will be an excess demand, which will be denoted by :χ_t.

where v_j are the value functions obtained in step 2. Unlike in the previous algorithm, expected equity premium is not included as the additional state variable.

(c) Depending on the realization for z' , compute the joint distribution of wealth, labor efficiency and employment status.

(d) To generate a long time series of the movement of the economy, repeat substeps b) and c).

4. Use the time series from step 3 and perform a regression of lnK' on constants and lnK, for all possible values of z. This way, the new aggregate laws of motion for capital are obtained.

However, now for the law of motion for the equity premium, we cannot run a regression, since we do not have "true" market clearing bond prices (equity premium). Instead, we have excess demand in each time-period, given the perceived equity premium. We can use this information to update the perceived law of motion for equity premium. To do this, the Broyden method (Broyden, 1965) is used:

Consider a system of equations

$$f(x^*) = 0$$

, where x are the "true" coefficients of the perceived law of motion for equity premium

$$x^* = (b_0^*(z), b_1^*(z))$$

and

$$f(x) = (f_1(b_0^*(z), b_1^*(z)), f_2(b_0^*(z), b_1^*(z)))$$

f_1 and f_2 denote the error measures that is chosen.[5] For this algorithm, I propose these two measures to be coefficients of a linear regression of excess demand on a constant and capital. The true solution to the model would have the coefficients of this regression equal to 0. This would mean that the mean value of excess demand is 0 and also that the excess demand do not depend on the amount of capital K. Therefore, to obtain f_1 and f_2, one has to run the following regressions:[6]

$$\xi_t(z) = \varrho_1(z) + \varrho_2(z)K_t + \epsilon_t$$

One can also use a linear coefficient, and instead of a coefficient on a constant to use an average excess demand for a given aggregate state. In this particular example this provides a faster convergence. After this, step error measures are obtained:

$$f_1(b_0^*(z), b_1^*(z)) = \varphi \sum \xi_t(z)$$

where φ is arbitrary constant.[7]

$$f_2(b_0^*(z), b_1^*(z)) = \varrho_2(z)$$

Now, the goal is to find the true x^*. This is conducted in the following steps:

(a) First, define $\chi^n = f(x^n)$. Where χ^n and x^n denote the excess demand measure and the coefficients in the iteration n.

$$\chi^n = (f_1(b_0(z), b_1(z)), f_2(b_0(z), b_1(z)))$$

Furthermore: $\Delta x_n = x_n - x_{n-1}$, $\Delta \chi_n = \chi_n - \chi_{n-1}$

[5] One particular error measure is proposed, but many others can be used, depending on the model and the convenience. For example, another one can be using a simple sum of excess demand in each period. Then, the sample would be partitioned into two, depending if the capital is higher or lower than a certain threshold. This would have to be done, as we need to determine two coefficients for each aggregate state. If, for example, the perceived law of motion would have a quadratic form, the sample would be partitioned into three partitions, depending on the level of capital, etc.

[6] In additional, if the perceived law of motion was quadratic, we would use a quadratic regression, since we would need to obtain three parameters for each realization of the aggregate state.

[7] Alternatively, it is possible to simply use $f_1(b_0^*(z), b_1^*(z)) = \varphi\varrho_1(z)$. φ is used only as a parameter that gives relative weight of the two error outputs.

(b) For the initial iteration, we guess the Jacobian matrix. For each additional iteration, we update the Jacobian matrix by:

$$J_n = J_{n-1} + \frac{\Delta\chi - J_{n-1}\Delta x_n}{||x||^2}\Delta x_n^T$$

after updating the matrix, we update the guess of the perceived law of motion for equity premium:

$$x_{n+1} = x_n - J_n^{-1} f(x_n)$$

We do these steps two times, for $z = good$ and $z = bad$.

5. Compare the laws of motion from step 4 and step 1. If they are almost identical and their predictive power is sufficiently good, the solution algorithm is completed. If not, make a new guess for the laws of motion, based on a linear combination of laws from steps 1 and 4. Then, proceed to step 2.

5 Performance comparison on an example model

To demonstrate the potential reduction in the computation speed of the discussed model, I solve the model described in section 2, both with the classical solution method (Krusell and Smith, 1997) and the proposed method from section 4. To compare the two algorithms, the parametrized model will be solved 20 times by the two algorithms, each time starting from the different initial perceived law of motions. The initial perceived law of motion is obtained as follows: Each parameter of the true laws of motion is randomly perturbed by a normally distributed shock with the standard deviation $\sigma = 0.01$. The size of the perturbation is large enough so that the initial guess is not too close to the solution, and not too large to cause all of the households to have a corner portfolio solution.[8] The stopping criterion for the perceived laws of motion for equity premium is that the excess demand of the bonds have to be on, average smaller, than 0.1% of the aggregate capital, without imposing the market-clearing.[9] When updating the laws of motion parameters, the weight of the new guess is always 1. This is only because, for this specific model, it happens to minimize the time for convergence. In

[8]This is important since taking the numerical derivative of excess demand may not behave properly. For details see the discussion in Section 6.

[9]If the market-clearing is imposed, at least in the last iteration, the excess demand will be orders of magnitudes smaller. For details, see the discussion in Section 6.

the value function iteration, 85 grid points are used in the individual wealth dimension, and 12 grid points are used in the aggregate capital dimension. Cubic splines are used to interpolate the values in between the grid points. The code is written in a FORTRAN 90 programming language and compiled using Intel Fortran Compiler. All the simulations are executed on a personal computer using Linux Mint 18 (64-bit) operating system, with Intel i7-67000 Central Processing Unit (4 cores and 8 threads), clocked at 2.60 GHz. I report both the number of iterations necessary obtain a solution (a convergence), and an overall run-tim.

5.1 Parametrization

The model is parametrized to a quarterly frequency. The choice of the main parameters are reported in Table 1.

Table 1: Parameters

Parameter	Symbol	Value
Risk aversion	α	10
Intertemporal elasticity of substitution	$\frac{1}{\rho}$	0.50
Discount factor	β	0.901
Expected depreciation rate	$E(\delta)$	0.033
Standard deviation of depreciation rate	$\sigma(\delta)$	$1.0E-4$
Leverage	λ	0.35
Average tax rate for funding social security	τ^{lss}	0.07
Borrowing constraint: bonds	κ^b	-0.23
Borrowing constraint: stocks	κ^s	0.00
Chance of not retiring	θ	0.994
Chance of not dying	υ	0.983

The TFP shocks and capital depreciation shocks are assumed to be perfectly correlated, and thus there are only two aggregate states *good*, where TFP is high and depreciation is low, and *bad*, where TFP is low and depreciation is high.

5.1.1 Idiosyncratic shocks

There are 5 possible idiosyncratic states in which the household can find itself (5 for each aggregate state). The labor productivity among the working-age employed households is governed by the transitional Markov matrix:

$$\Pi_l = \begin{bmatrix} 0.9850 & 0.0100 & 0.0050 \\ 0.0025 & 0.9850 & 0.0125 \\ 0.0050 & 0.0100 & 0.9850 \end{bmatrix}$$

and for the individual labor productivity levels, the following values are used: $l \in \{36.5, 9.5, 1.2\}$. In addition to this risk, the households face a risk of becoming unemployed, which is the same regardless of the labor productivity level. Finally, working-age households also face a risk of becoming retired $1-\theta$. The average unemployment spell is set to 1.5 quarters in the good state (boom) and 2.5 quarters in the bad state (recession). The replacement rate for the unemployed is set to 4.2% of the average wage in the given period. The probabilities of becoming/remaining unemployed when the economy moves from a good to bad state and vice-versa is adjusted to match the movement of the overall employment, which is set to 95.9% in the good state and 92.8% in the bad state.

5.1.2 Generated Moments Appendix

The selection of the moments in the model is presented in Table 2.

Table 2: Moments in the model

Moment	Symbol	Value
Capital-output ratio	K/Y	7.01
Average interest rate	r^b	1.43 %
Expected return to capital	$E\{r^s\}$	1.44 %
Average equity premium	$E\{r^s - r^b\}$	0.01 %

5.2 Solution for perceived laws of motions

$$lnK' = a_0(z,\delta) + a_1(z,\delta)lnK$$

$$lnP^e = b_0(z,\delta) + b_1(z,\delta)lnK'$$

For the example model, the perceived aggregate law of motions are:

In a good TFP and δ state:

$$lnK' = 0.113 + 0.936lnK$$

$$lnP^e = -8.800 - 0.629lnK'$$

In a bad TFP and δ state:

$$lnK' = 0.111 + 0.934lnK$$

$$lnP^e = -8.100 - 0.407lnK'$$

The perceived laws of motion predict the actual movements of capital and equity premium with $R^2 = 0.99995$ for capital and $R^2 = 0.99999$ for equity premium.

The average error for the aggregate capital law of motion is 0.0026% percent of the capital stock, while the maximum error is 0.0110% of the capital stock.

5.3 Comparison

Table 3: Algorithm execution comparisons (including the obtaining of derivatives)

Algorithm	Average Iterations	Average run-time
Krusell and Smith (1997)	3.2	26 min. 18 sec.
Proposed algorithm	9.8	17 min. 59 sec.

The use of the proposed algorithm leads to a reduction in the run-time of 32%. The execution performance of the proposed model is measured conservatively, since taking the numerical derivatives to construct the initial Jacobian matrix is considered. Alternatively, if one has a reasonably good guess for the Jabocian matrix (perhaps from the previous simulations of the model with similar parameters), it can be guessed directly, without taking the numeric derivative. If the initial Jacobian was guessed, instead of computed, then the proposed algorithm would take 2 iterations less, and lead to a 46% reduction in run-time.

Table 4: Bond market errors: absolute average excess demand in terms of percentage of aggregate capital)

Algorithm	Before imposing market clearing	After imposing market clearing
Krusell and Smith (1997)	0.0949%	0.0021%
Proposed algorithm	0.0950%	0.0019%

After obtaining final laws of motion, the simulation of the model is run with clearing of the bond market in each time period (like in the classical version of the algorithm). This is to compare and show that the obtained laws of motion are of approximately the same accuracy (they are basically approximately identical). In terms of R^2, the proposed algorithm generates R^2 of 0.99995418 for capital and 0.99999712 for equity premium, while the classical version of the algorithm generates R^2 of 0.99995418 for capital and 0.99999733. Both by looking at the R^2 and Table 4, one can see that the laws of motion produce almost identical results.

6 Discussion

The main reason for the computational speed-up in the proposed algorithm is avoiding root-finding (finding the bond market-clearing price) for each simulated period t. However, the proposed algorithm takes more iterations to converge to the true solution. Therefore, the proposed algorithm is able to perform each iteration much faster (on average four and a half times faster), but takes more iterations to converge (on average three times more). However, the speed-up coming from a faster simulation of the economy outweighs the increased number of iterations, which leads to a reduction in total run-time.

The reported speed-up due to the proposed algorithm is conservative. The reason is twofold. First, the reported time and number of iterations includes numerically taking derivatives used to construct the initial guess for the Jacobian matrix J. If one would have a reasonably good guess for the Jacobian, which is often the case if the changes in parameters are small compared to the previously computed model, then it is possible to avoid the first two iterations of the proposed algorithm. For example, if the values of initial Jacobian were guessed, instead of computed, the proposed algorithm would take 2 iterations less and would lead to a 46% reduction in run-time. The second reason is that the initial guess for the Value function computation stage is always the same, and it is the value of consuming the entire wealth in one period. An alternative option would be to use the value function from the previous iteration as the initial guess for the value function for the current iteration. The choice is also biased towards the classical algorithm from Krusell and Smith (1997), since the proposed algorithm performs more iterations and Value function iterations to converge. Using better (circumstantial) initial value function guesses would decrease the speed-up from the proposed algorithm even more (for example: final guesses from previous iterations).

As mentioned in section 5, all the initial guesses for the laws of motion are such that at least some households have an internal portfolio choice. This is to ensure that the derivative of excess demand with respect to perceived equity premium would not be zero. This condition is important when constructing the initial Jacobian matrix in the proposed algorithm. If the condition is not satisfied, this does not mean that the proposed algorithm cannot be used. One can simply use the classical version of the algorithm until the condition is satisfied, and then continue updating using the proposed version of the algorithm.[10]

Furthermore, the threshold for the excess demand caused by using the predicted equity premium is 0.1% (on average).[11] This is true for both the classical and the proposed versions of the algorithm. However, the actual excess demand are orders of magnitudes smaller in the classical algorithm, because the classical algorithm imposes the bond market-clearing each period, and the equity premium is then not restricted by the (linear) shape of the perceived laws of motion. However, this should not be perceived as a disadvantage of the proposed algorithm. One can see it only as a way to arrive at the true laws of motion, and then when the correct perceived laws of motion are computed, in the last iteration approximately exact market-clearing can be imposed.

The proposed version of the algorithm is particularly useful in asset pricing models with uninsurable idiosyncratic and aggregate risk. This is because the perturbation methods in the style of Reiter (2009) are not precise when applied to these types of models, as they assume linearity in the aggregate states (Reiter, 2009). To this date, the usual method for solving these types of models are variations of the algorithm described in Krusell and Smith (1997). The proposed algorithm can be used to improve on the classical Krusell-Smith algorithm whenever a market-clearing has to be imposed explicitly,[12] such as models with endogenous labor supply (although one might opt not to use Krusell-Smith algorithms at all).

[10]For similar reasons, the proposed version of the algorithm tends to perform better when the guess for the equity premium laws of motion are relatively good, and perceived laws of motion for aggregate capital are relatively bad, the classical version of the algorithm tends to do better if the opposite is true.

[11]This may seem like a large value, but the changes in the equity premium producing such excess demand are very small, also by measuring how much they impact the welfare of the agents.

[12]The computation of the model without portfolio choice (Krusell and Smith, 1998) likely cannot be improved using the proposed algorithm, as in the case with only one good the market clears by Warlas's law. Therefore, allowing non-clearing of the markets would be superfluous, as we can clear it directly from the budget constraint. One might use a Newton-like method to update the laws of motion for capital, instead of using the regression. However, this will probably require more iterations to arrive at the solution. One can see this in Table 3, where the proposed algorithm takes more iterations to arrive at the solution. The time savings come from not clearing the bond market in each time period t, and thus performing each iteration is shorter.

7 Conclusion

This paper shows how to reduce the run-time of the Krusell-Smith algorithm (Krusell and Smith, 1997) by proposing an alternative version of the algorithm. The reduction in computation time is achieved by avoiding the computationally expensive root-finding procedure to clear the bond markets in every simulated period while finding the correct perceived laws of motion. Instead, the proposed algorithm lets the economy proceed with the uncleared bond markets, and uses the information on the excess demand to update the perceived laws of motion. The guesses on the perceived laws of motion are updated using the Newton-like method described in Broyden (1965).

Measured conservatively, the proposed algorithm leads to a decrease in computation time of 32% in the example model. By using better circumstantial initial guesses on the value function and initial Jacobian matrix, the computational improvement would be even higher.

The described algorithm is useful in reducing the computational time of asset pricing models with uninsurable idiosyncratic and aggregate risk, although it can be used in other models that require market-clearing to be explicitly imposed.

References

Algan, Y., Allais, O., and Carceles-Poveda, E. (2009). Macroeconomic Effects of Financial Policy. *Review of Economic Dynamics*, 12(4):678–696.

Broyden, C. (1965). A class of methods for solving nonlinear simultaneous equations. (19 (92): 577-593.).

Carroll, C. D. (2006). The method of endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics Letters*, 91(3):312–320.

Den Haan, W. J. (1997). Solving Dynamic Models With Aggregate Shocks And Heterogeneous Agents. *Macroeconomic Dynamics*, 1(02):355–386.

Epstein, L. G. and Zin, S. E. (1989). Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica*, 57(4):937–969.

Krueger, D., Mitman, K., and Perri, F. (2016). *Macroeconomics and Household Heterogeneity*, volume 2 of *Handbook of Macroeconomics*, chapter 12, pages 843–921. Elsevier.

Krusell, P. and Smith, A. A. (1997). Income And Wealth Heterogeneity, Portfolio Choice, And Equilibrium Asset Returns. *Macroeconomic Dynamics*, 1(02):387–422.

Krusell, P. and Smith, A. A. (1998). Income and Wealth Heterogeneity in the Macroeconomy. *Journal of Political Economy*, 106(5):867–896.

Reiter, M. (2002). Recursive Solution Of Heterogeneous Agent Models. Manunscript, Universitat Pompeu Fabra.

Reiter, M. (2009). Solving heterogeneous-agent models by projection and perturbation. *Journal of Economic Dynamics and Control*, 33(3):649–665.

Walras, L. (1874). *Eléments d'économie politique pure.*

Walras, L. (1954). *Elements of Pure Economics or the Theory of Social Wealth.* Number Translated by W. Jaffé.

Young, E. R. (2010). Solving the incomplete markets model with aggregate uncertainty using the Krusell-Smith algorithm and non-stochastic simulations. *Journal of Economic Dynamics and Control*, 34(1):36–41.